ENGINEERING TRIPOS PART IIB

Mon 2 May 2005 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 Let $\{x(n)\}$ be a sequence of independent and identically distributed symbols such that

$$\Pr\{x(n)=1\}=\Pr\{x(n)=-1\}=0.5.$$

These symbols are transmitted through a communication channel with output

$$y(n)=\sum_{k=0}^{L-1}\alpha_k x(n-k)+v(n)$$

where $\{v(n)\}$ is zero-mean white noise of variance $E\left[v(n)^2\right]=\sigma_v^2$ and is statistically independent of $\{x(n)\}$.

(a) Determine the Wiener filter $\mathbf{h}_{\mathrm{opt}}$ which minimizes

$$J(\mathbf{h})=E\left[\left(x(n)-\mathbf{h}^{\mathrm{T}}\mathbf{y}(n)\right)^2\right]$$

as a function of

$$\mathbf{R}=E\left[\mathbf{y}(n)\mathbf{y}^{\mathrm{T}}(n)\right],$$
$$\mathbf{p}=E\left[\mathbf{y}(n)x(n)\right],$$

where

$$\mathbf{y}(n)=(y(n),\ y(n+1),\ \cdots,\ y(n+M-1))^{\mathrm{T}}$$

Your solution should include explicit expressions for $\mathbf{R}$ and $\mathbf{p}$. [50%]

(b) The value of $J(\mathbf{h}_{\mathrm{opt}})$ is a function of the length M of the Wiener filter. Without any calculation, explain why there exists a value K such that $J(\mathbf{h}_{\mathrm{opt}})$ is constant for $M \geq K$. What is this value of K? [25%]

(c) In many practical situations the Wiener filter cannot be computed as it relies on the coefficients $\{\alpha_k\}$ and σ_v^2, which are unknown quantities. Using a training sequence of symbols $\{x(n)\}$, describe how you would apply the LMS algorithm in this context to approximate the Wiener filter. What are the advantages and disadvantages of such an approach? [25%]

2 (a) Consider the following signals

$$\begin{aligned} &L\text{-tap FIR Filter,} \quad u(n) = \textstyle\sum_{k=0}^{L-1} \beta_k w(n-k) \\ &2\text{–tap IIR Filter,} \quad u(n) = \alpha u(n-1) + w(n) \end{aligned}$$

where $|\alpha| < 1$ and $\{w(n)\}$ is zero-mean white noise of variance $E\left[w(n)^2\right] = \sigma^2$.

If these signals are the input to an LMS filter of length M, the stability limit on the stepsize μ is given by $\left(M\,E\left[u(n)^2\right]\right)^{-1}$. Calculate this limit for these two signals. [30%]

(b) Let $\{u(n)\}$ be a zero-mean input signal and $\{d(n)\}$ be a reference signal. Consider the following recursive algorithm

$$\mathbf{h}(n) = (1 - \mu\gamma)\,\mathbf{h}(n-1) + \mu\mathbf{u}(n)\,e(n) \qquad (1)$$

where

$$e(n) = d(n) - \mathbf{h}^{\mathrm{T}}(n-1)\,\mathbf{u}(n)$$

with $\gamma > 0$, $\mathbf{u}(n) = (u(n),\ u(n-1), \cdots,\ u(n-M+1))^{\mathrm{T}}$ and $\mathbf{h}(n) = (h(n),\ h(n-1), \cdots,\ h(n-M+1))^{\mathrm{T}}$.

Assuming the expectation of $\mathbf{h}(n)$, denoted $E[\mathbf{h}(n)]$, converges towards a limit $\mathbf{h}$, determine this limit by making the following standard approximation:

$$E\left[\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\,\mathbf{h}(n-1)\right] \approx E\left[\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right] E\left[\mathbf{h}(n-1)\right].$$

Express the result as a function of

$$\begin{aligned} \mathbf{R} &= E\left[\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right], \\ \mathbf{p} &= E\left[\mathbf{u}(n)\,d(n)\right]. \end{aligned}$$

[40%]

(c) An approximate analysis of the recursion (1) shows that it is numerically stable if

$$\mu \le \frac{2}{\lambda_{\max} + \gamma}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{R}$. In practice this eigenvalue cannot be computed exactly. Propose and justify an alternative criterion based on $E\left[u(n)^2\right]$ which ensures the stability of (1). How would you approximate $E\left[u(n)^2\right]$ in a real-world scenario? [30%]

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3 (a) The periodogram estimate for power spectrum estimation can be written as

$$\hat{S}_X(e^{j\omega T}) = \sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega T}$$

where N is the length of the data. Discuss briefly the properties of the periodogram, including bias, variance, frequency resolution and computation. [20%]

Show that when the *biased* estimator for autocorrelation is applied, the periodogram can be determined directly from the DTFT of a windowed version of the original data,

$$\hat{S}_X(e^{j\omega T}) = \frac{1}{N}|X_w(e^{j\omega T})|^2$$

where $X_w(e^{j\omega T})$ should be defined appropriately. [30%]

(b) A moving average random process is defined as:

$$x_n = 0.9e_n + 0.1e_{n-1}$$

where $\{e_n\}$ is zero-mean white noise having unity variance, $E[e_n^2] = 1$. Determine the power spectrum of this process. [20%]

The biased method of autocorrelation estimation is employed. Use the result from part a) to show that the expected value of the corresponding periodogram estimate of the power spectrum is given by

$$E[\hat{S}_X(e^{j\omega T})] = 0.82 + 0.18\frac{N-1}{N}\cos(\omega T)$$

Is the periodogram estimate biased for this MA process? Is it biased asymptotically (i.e. as the data size becomes very large)? Justify your answer. [30%]

4 (a) An ARMA model has the following difference equation:

$$x_n = -\sum_{p=1}^{P} a_p\, x_{n-p} + \sum_{q=0}^{Q} b_q\, w_{n-q}$$

Discuss how the ARMA model can be used for power spectrum estimation. You should include the formula for the power spectrum of the ARMA model, as well as a discussion of its advantages/disadvantages compared with non-parametric procedures. [35%]

(b) Show that the ARMA model autocorrelation function obeys the following difference equation

$$R_{XX}[r] + \sum_{p=1}^{P} a_p\, R_{XX}[r-p] = \begin{cases} c_r & \text{if } r \leq Q \\ 0, & \text{if } r > Q \end{cases}$$

where:

$$c_r = \sum_{q=r}^{Q} b_q\, h_{q-r}$$

and h_n is the impulse response of the corresponding IIR filter. [35%]

(c) Four values from the autocorrelation function of an ARMA model with $P = 2$ and $Q = 1$ are given by

$$R_{XX}[0] = 1.84,\ \ R_{XX}[1] = 1.32,\ \ R_{XX}[2] = 0.75,\ \ R_{XX}[3] = 0.47$$

Use the result of part b) to set up and solve the equations for the AR coefficients a_1 and a_2 in this ARMA model. [30%]

END OF PAPER