UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB

Monday 25 April 2005 9 to 10.30

Module 4F8

IMAGE PROCESSING AND IMAGE CODING

Answer not more than two questions.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) A continuous image $g(u_1, u_2)$ is converted to an array of samples $g_s(u_1, u_2)$. For a rectangular sampling grid with spacings Δ_1 and Δ_2 in the u_1 and u_2 directions, write down the 2-dimensional Fourier transform (spectrum), $G_s(\omega_1, \omega_2)$, of the sampled signal in terms of the Fourier transform, $G(\omega_1, \omega_2)$, of the original image. Using this expression for $G_s(\omega_1, \omega_2)$ explain the phenomenon of *aliasing*. [25%]

(b) The image $g(u_1, u_2)$ is sampled on the non-standard hexagonal lattice, shown in Fig. 1, to produce a sampled image $g_s(u_1, u_2)$. If each regular hexagon has side s, write down an expression for this array as the sum of two sampling functions, s_1 for the grid marked by squares, and s_2 for the grid marked by circles in Fig. 1.

If s_1 and s_2 have Fourier coefficients c_1 and c_2, where

$$c_1(p_1, p_2) = \frac{1}{2\sqrt{3}\, s^2} \quad \text{and} \quad c_2(p_1, p_2) = \frac{1}{\sqrt{3}\, s^2} \exp\left(-j(p_1 + p_2)\pi\right)$$

show that the spectrum $G_s(\omega_1, \omega_2)$ of the sampled signal takes the following form

$$G_s(\omega_1, \omega_2) = \sum_{p_2=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} G\left(\omega_1 - \frac{\pi p_1}{s},\ \omega_2 - \frac{2\pi p_2}{\sqrt{3}\, s}\right) f(p_1, p_2)$$

and calculate the function $f(p_1, p_2)$. [60%]

(c) Sketch a contour plot to explain the effect of this sampling pattern on the original spectrum $G(\omega_1, \omega_2)$. (Assume any form for $G(\omega_1, \omega_2)$ and also that the sampling is above the Nyquist frequencies). [15%]

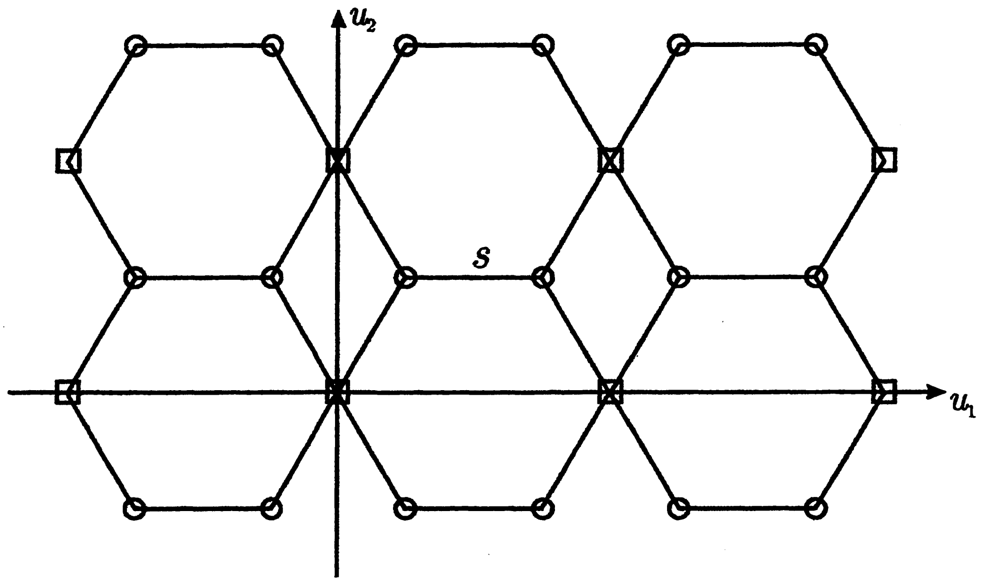

Fig. 1

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2 (a) An ideal 2-dimensional lowpass filter has a rectangular passband which extends from $-\Omega_1$ to Ω_1 in the ω_1 direction and from $-\Omega_2$ to Ω_2 in the ω_2 direction. Sampling is done on a rectangular grid with spacings Δ_1 and Δ_2 in the u_1 and u_2 directions (assume $\Omega_1 < \pi/\Delta_1$ and $\Omega_2 < \pi/\Delta_2$). Show that the ideal impulse response, $h(n_1, n_2)$, of this lowpass filter is given by

$$h(n_1, n_2) = \frac{\Delta_1 \Delta_2}{\pi^2} \, \Omega_1 \, \Omega_2 \, \text{sinc}(\Omega_1 n_1 \Delta_1) \, \text{sinc}(\Omega_2 n_2 \Delta_2)$$ [30%]

Now suppose we have an ideal bandpass filter as shown in Fig. 2, where the shaded regions are the passbands of the filter and the sampling is as specified above. By viewing this ideal frequency response as a linear combination of lowpass filter responses or of more standard bandpass responses, calculate the ideal impulse response, $h(n_1, n_2)$, of this bandpass filter. [30%]

(b) The process of *histogram equalisation* is the application of a transformation to an input image x to produce an output image y via $y = g(x)$, in such a way that the probability of the occurrence of various grey levels is constant. If X and Y are random variables associated with the input and output, the probability of X lying between x and $x + \delta x$ is the same as the probability of Y lying between y and $y + \delta y$ (for $\delta x, \delta y$ small), which implies that

$$p_Y(y) \, dy = p_X(x) \, dx$$

If we require $p_Y(y)$ to take a constant value of $1/L$ in the range 0 to L, derive an expression for y_k, the k^{th} mapped grey level, in terms of N_i, the number of pixels in bin x_i to $x_i + \Delta x_i$ of the input histogram. Assume that (N, M) are the dimensions of the image and that k runs from 0 to $N_L - 1$ (i.e. there are a total of N_L grey levels). State carefully any assumptions and approximations that are made in the derivation. [40%]

(cont.

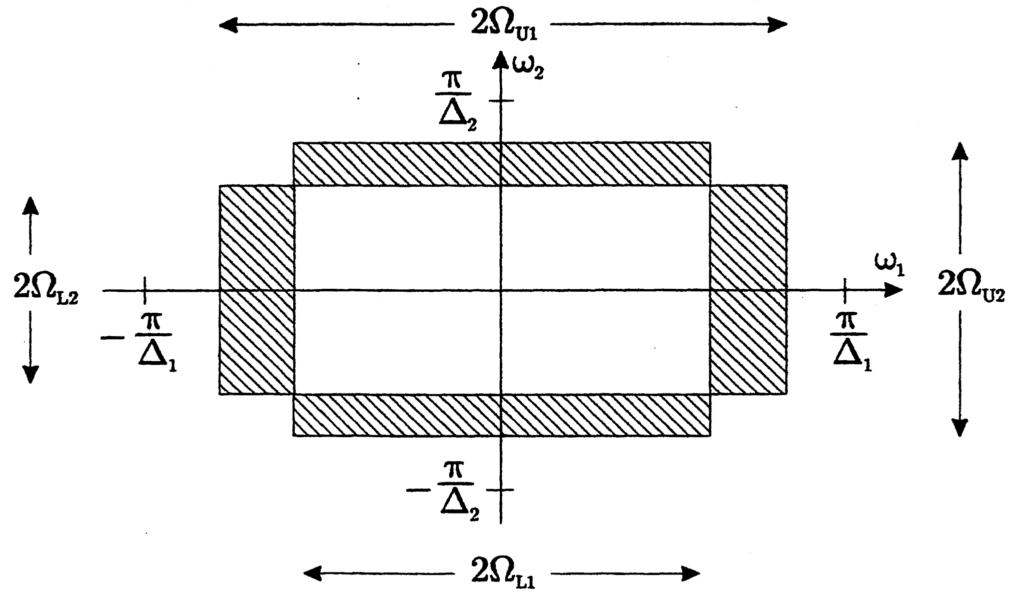

Fig. 2

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3 (a) If $\mathbf{T}$ is an orthonormal $n \times n$ discrete cosine transform (DCT) matrix, derive the operation which must be applied to an $n \times n$ matrix of DCT coefficients $\mathbf{Y}$ in order to reconstruct an $n \times n$ image region $\mathbf{X}$. [20%]

(b) Explain how this inverse DCT may be expressed in general as the weighted sum of n^2 2-dimensional basis functions, each of size $n \times n$, and obtain a general expression for each basis function in terms of rows of the matrix $\mathbf{T}$. [25%]

(c) For a large monochrome input image of size $kn \times kn$ pixels, the subband $\mathbf{S}_{i,j}$, of size $k \times k$, is formed from the elements $y_{i,j}$ of each of the matrices $\mathbf{Y}$ of DCT coefficients. For a typical input image, the coefficient values of the subbands $\mathbf{S}_{i,j}$ are found to have probability density functions which are Laplacian and the standard deviations of these are given by

$$\sigma_{i,j} = \frac{A}{i+j} \qquad \text{for } i = 1 \ldots n,\ j = 1 \ldots n$$

Assuming that the entropy of samples from a Laplacian distribution of standard deviation σ, quantised with a step size of Q, can be approximated by

$$H(\sigma, Q) = \tfrac{1}{2} \log_2 \left(\frac{2\, e^2\, \sigma^2}{Q^2} + 1 \right)$$

estimate the number of bits needed to code the above large image, when $n = 4$, $k = 256$ and $Q = A/2$. [35%]

(d) Explain why, in a practical image coder, the DCT coefficients in higher frequency subbands can usually be quantised more coarsely than lower frequency ones and why the provision of colour information typically needs an increase of less than 50% in the number of bits to code a given image. [20%]

4 (a) Figure 3 shows three equivalent arrangements of a filter with down-sampling and up-sampling operations. Sketch a 3-level wavelet filter analysis tree for 1-dimensional signals and then show how the filters and downsamplers may be rearranged to allow simple calculation of overall z-transfer functions from the input to any of the tree outputs. [30%]

(b) If the lowpass and highpass filters in the original wavelet tree of part (a) are $H_0(z)$ and $H_1(z)$, calculate the z-transfer functions from the input to each of the two outputs at level m of an m-level wavelet tree. [20%]

(c) The (3,5)-tap filters for a wavelet tree are given by

$$H_0(z) = \tfrac{1}{4}(z + 2 + z^{-1}) \qquad \text{and} \qquad H_1(z) = \tfrac{1}{8}(-z - 2 + 6z^{-1} - 2z^{-2} - z^{-3})$$

Calculate the transfer functions to both level-2 outputs of a 1-dimensional 2-level tree which uses these filters. [20%]

(d) The above pair of filters is now used in a 2-dimensional 2-level wavelet tree. Explain how a matrix representing the 2-dimensional impulse response (or point-spread function) of a level-2 subband may be calculated from the 1-dimensional responses of its constituent filters, and sketch the impulse response of the level-2 Hi-Lo subband, in which the highpass response applies to the rows of the input image and the lowpass response applies to the columns. To what type of simple image feature will this subband respond strongly? [30%]

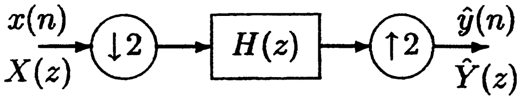

Fig. 3a.

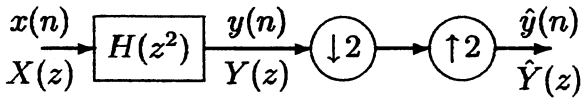

Fig. 3b.

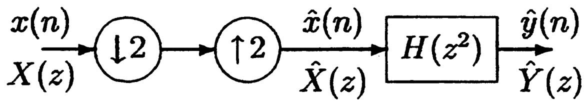

Fig. 3c.

END OF PAPER

Module 4F8, April 2005 – IMAGE PROCESSING AND IMAGE CODING – Answers

1(a) $$G_s(\omega_1, \omega_2) = \frac{1}{\Delta_1 \Delta_2} \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} G(\omega_1 - p_1\Omega_1, \omega_2 - p_2\Omega_2)$$

where $\Omega_i = \frac{2\pi}{\Delta_i}, i = 1, 2.$

1(b) $$f(p_1, p_2) = \frac{1}{2\sqrt{3}\, s^2} \left[1 + 2\alpha \exp\left(-j(\frac{p_1}{2} + p_2)\pi\right)\right]$$

where $\alpha = 0$ if p_1 is odd and $\alpha = 1$ if p_1 is even.

2(a) $$\begin{aligned} h(n_1, n_2) &= \frac{\Delta_1\Delta_2}{\pi^2} [\Omega_{U1}\, \Omega_{L2} \operatorname{sinc}(\Omega_{U1} n_1 \Delta_1) \operatorname{sinc}(\Omega_{L2} n_2 \Delta_2) \\ &\quad + \Omega_{L1}\, \Omega_{U2} \operatorname{sinc}(\Omega_{L1} n_1 \Delta_1) \operatorname{sinc}(\Omega_{U2} n_2 \Delta_2) \\ &\quad - 2\, \Omega_{L1}\, \Omega_{L2} \operatorname{sinc}(\Omega_{L1} n_1 \Delta_1) \operatorname{sinc}(\Omega_{L2} n_2 \Delta_2)] \end{aligned}$$

2(b) $$y_k = \sum_{i=0}^{k} L \frac{N_i}{NM}$$

3(a) $$\mathbf{X} = \mathbf{T}^T\, \mathbf{Y}\, \mathbf{T}$$

3(b) $$\mathbf{T}(i, j) = \mathbf{t}_i^T\, \mathbf{t}_j \quad \text{so that } \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} y_{i,j}\, \mathbf{T}(i, j)$$

3(c) $$N_{\text{bits}} = 990.28 \text{ Kbit} = 1014048 \text{ bits}$$

4(b) $$H_{0\cdots00} = \prod_{k=1}^{m} H_0(z^{2^{k-1}}) \quad \text{and} \quad H_{0\cdots01} = \left[\prod_{k=1}^{m-1} H_0(z^{2^{k-1}})\right] H_1(z^{2^{m-1}})$$

4(c) $$\begin{aligned} H_{00} &= \tfrac{1}{16}(z^3 + 2z^2 + 3z + 4 + 3z^{-1} + 2z^{-2} + z^{-3}) \\ H_{01} &= \tfrac{1}{32}(-z^3 - 2z^2 - 3z - 4 + 4z^{-1} + 12z^{-2} \\ &\quad + 4z^{-3} - 4z^{-4} - 3z^{-5} - 2z^{-6} - z^{-7}) \end{aligned}$$