ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Friday 6 May 2005 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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1 The first order wave equation $u_t + Cu_x = 0$; $C > 0$; $t > 0$; $x > 0$; can be solved numerically by discretising the equation with the "Forward Time–Backward Space" (FTBS) algorithm:

$$u_i^{n+1} = u_i^n - \frac{C\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

where Δt is the increment of the discretisation in time and Δx the increment along the x axis in space, as shown in Fig. 1 below.

(a) Discuss initial and boundary conditions needed in order to obtain a meaningful solution. [25%]

(b) Show that to second-order accuracy, the discretised equation actually represents a different PDE:

$$u_t + Cu_x - \kappa u_{xx} = 0; \quad \kappa = \frac{C\Delta x}{2}(1 - \frac{C\Delta t}{\Delta x}).$$ [25%]

(c) Based on the result of (b), discuss the impact of the discretisation on the solution of the PDE, and find the condition under which the numerical solution becomes exact. [25%]

(d) Discuss the well-posedness of the new PDE $u_t + Cu_x - \kappa u_{xx} = 0$ with the original initial and boundary conditions that you chose for $u_t + Cu_x = 0$ in (a). [25%]

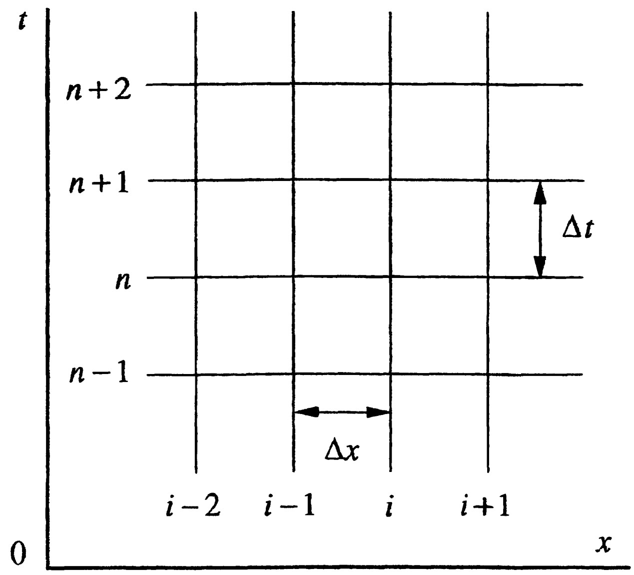

Figure 1

2 A potential problem is defined in an annular domain:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0; \;\; 1 < r < 2.$$

(a) Solve the problem with Dirichlet boundary conditions:

$$u(1,\theta) = 5 \quad \text{and} \quad u(2,\theta) = 8, \quad 0 \le \theta \le 2\pi.$$ [25%]

(b) Verify that the Maximum Principle for Elliptic Equations is valid for this case:

$$5 < u(r,\theta) < 8, \quad 1 < r < 2.$$ [25%]

(c) Using the Maximum Principle, show that the solution you obtained is unique and stable, and thus show that the Dirichlet problem is well-posed. [25%]

(d) If Neumann boundary conditions are used instead of Dirichlet boundary conditions for the same potential field:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2,$$

$$\frac{\partial u(1,\theta)}{\partial r} = C_1 \quad \text{and} \quad \frac{\partial u(2,\theta)}{\partial r} = C_2, \quad 0 \le \theta \le 2\pi,$$

where C_1 and C_2 are constants, what constraint should be imposed on the conditions at the two boundaries in order to maintain the well-posedness of the problem? [25%]

3 (a) (i) Use suffix notation to find an alternative expression for $\mathbf{u}\times(\nabla\times\mathbf{u})$, where $\mathbf{u}$ is a vector function of position. [15%]

(ii) An inviscid, incompressible fluid has a constant density ρ and a velocity field $\mathbf{u}$ which satisfies both

$$\nabla\cdot\mathbf{u}=0$$

and the Navier-Stokes equation

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]+\rho\nabla p+\nabla\psi=0$$

where p is the pressure and ψ is the gravitational potential energy. Use the result of (i) to show, using suffix notation, that

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}=0$$

where

$$\boldsymbol{\omega}=\nabla\times\mathbf{u}$$

is the vorticity of the flow field. [35%]

(b) (i) Stokes' theorem in its standard form states that

$$\iint_R \nabla\times\mathbf{f}\cdot d\mathbf{A}=\oint_C \mathbf{f}\cdot d\mathbf{l}$$

where $\mathbf{f}$ is a vector field, and R is a surface region with a boundary curve C. Show how this can be modified to a form which applies to a scalar field ϕ in place of the vector field $\mathbf{f}$. [15%]

(ii) If the scalar field ϕ is a function of x and y only, independent of z, and R is a region of the plane $z = 0$ bounded by a curve C, show that

$$\iint_R \nabla\phi\,dx\,dy=-\mathbf{n}\times\oint_C \phi\,d\mathbf{l}$$

where $\mathbf{n}$ is the vector (0,0,1). [35%]

4 (a) Figure 2 shows a sketch of a suspension bridge. A horizontal bridge deck is supported by many vertical cables suspended from an inextensible main cable of length L, in such a way that there is no bending moment or shear force in the deck. The main cable hangs from two fixed columns of equal height separated by a horizontal distance $2d < L$. The deck has mass m per unit length. The mass of the suspension cables can be neglected.

Show that the main suspension cable has a shape $u(x)$ which maximises the integral

$$\int_{-d}^{d} mgu \, dx$$

subject to the constraint on total length of the cable, where u is measured downwards from a horizontal line joining the tops of the columns, horizontal distance x is measured from the centre of the bridge, and g is the acceleration due to gravity. [25%]

(b) Find a differential equation which governs the shape $u(x)$, and solve it to show that the shape is an arc of a circle. [75%]

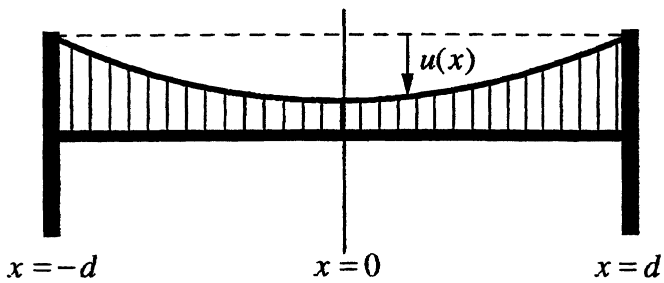

Figure 2

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Answers

2 (a) $u(r,\theta) = \dfrac{3}{\ln 2}\ln r + 5$

3 (a) $\dfrac{1}{2}\nabla(\mathbf{u}.\mathbf{u}) - (\mathbf{u}.\nabla)\mathbf{u}$