UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIB
ENGINEERING TRIPOS PART IIA

Thursday 28 April 2005 9 to 10.30

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

Answer not more than ***three*** *questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4M13 datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

1 (a) Identify the location and type of all singularities of the following functions:

(i) $f(z)=\dfrac{z\sqrt{z-1}}{\sin z}$

(ii) $f(z)=(z+1)^{-3}\sqrt{z^2+1}$.

Calculate the residue at any poles, and indicate a possible choice of branch cuts where branch points exist. [40%]

(b) Evaluate the following integral by contour integration

$$I=\int_0^{\infty}\frac{x^{1/3}}{x^2+1}dx.$$ [60%]

2 (a) Calculate the Laurent series expansion up to the constant term about each of the poles for the function

$$f(z)=\frac{\cos z}{z^2[z-2]}.$$ [30%]

(b) The Fourier Transform $F(\omega)$ of the function $f(x)$ can be defined by

$$F(\omega)=\int_{-\infty}^{\infty}f(x)e^{i\omega x}dx$$

with inverse

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega x}d\omega.$$

Given the choice

$$f(x)=0, \qquad x\le 0$$
$$f(x)=e^{-ax}, \qquad x>0$$

where $a>0$, determine the forward transform $F(\omega)$, and then take the inverse transform to show that $f(x)$ is recovered. [70%]

SECTION B

3 A fuel supplier produces a range of fuels for various applications (automotive, agricultural, domestic etc.) by blending two components, regular and octane. There are four grades in the range with the following blends and prices:

Grade	Regular (%)	Octane (%)	Price (£/litre)
1	100	0	0.30
2	90	10	0.34
3	80	20	0.37
4	70	30	0.39

The supplier has 64,000 litres of regular and 12,000 litres of octane, and wants to maximize income from this stock.

(a) Assuming the supplier can sell all of any of the grades, set up a linear programming problem to identify the balance of grades that maximizes income. [10%]

(b) Given that a feasible initial solution is $\underline{x} = (36\times10^3, 0, 0, 40\times10^3)$ where x_i is the quantity of grade i to be produced, show that the initial Simplex tableau for this problem in *canonical form* is

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 0 & 36\times10^3 \\ 0 & 1/3 & 2/3 & 1 & 40\times10^3 \\ 0 & -0.01 & -0.01 & 0 & 26.4\times10^3 \end{bmatrix}$$

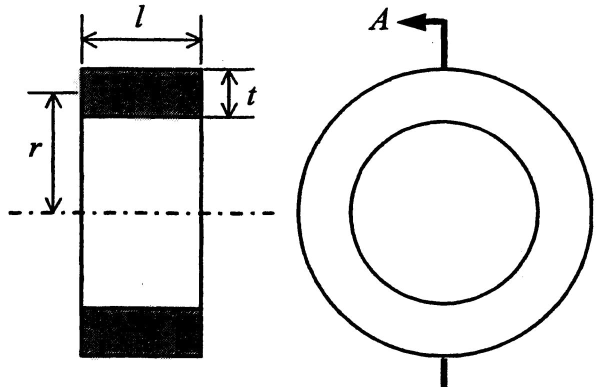
and hence use phase 2 of the Simplex method to find the optimal balance of fuel production. [70%]

(c) Explain how phase 1 of the Simplex method can be used to find an initial feasible solution to this linear programming problem. [20%]

4 The inductance I in microhenries of a randomly wound air-core inductance coil of the geometry shown in Fig. 1 is given by

$$I = \frac{31.5r^2n^2}{6r + 9l + 10t},$$

where r is the average coil radius and n is the number of coils contained within the area $l \times t$. Here r, l and t are measured in m.



Section *A-A*

Figure 1

It is desired to maximize the inductance of such a coil for a fixed length of wire L and a fixed occupied volume V, so that $2\pi rn = L$ and $2\pi rlt = V$.

(a) Formulate this design task as a *minimization* problem in four control variables (r, l, t and n) with two equality constraints. [10%]

(b) Use the Lagrange multiplier method to show that at the optimum $r = 1.5l$ and $t = 0.9l$. Hence find the optimal configuration (the optimal values of r, l, t and n and the consequent value of I) when $V = \pi / 7290$ m^3 and $L = 2\pi$ m. There is no need to check the second-order optimality conditions. [70%]

(c) How does the optimal inductance change if, separately, V and L are increased by 1%? [20%]

END OF PAPER

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OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*) f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

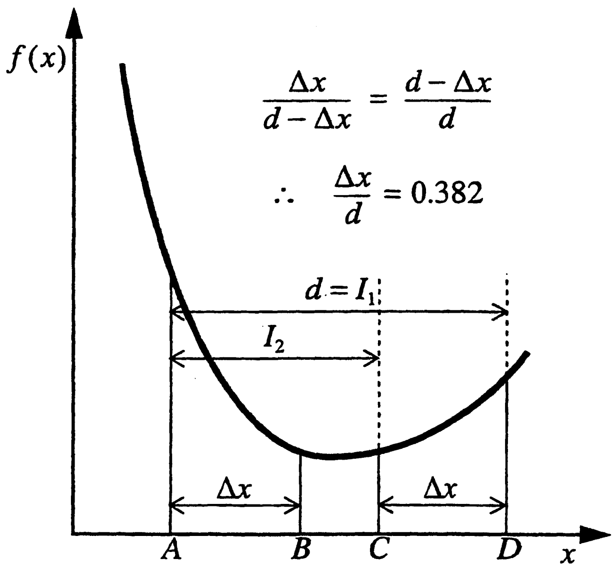
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1} \nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T \mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k) \mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T \mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\, \mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|} \right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k \mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T \nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\, \mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T \mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\, J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1} J(\mathbf{x}_k)^T \mathbf{r}(\mathbf{x}_k)$

$$\text{where } \boldsymbol{J}(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0$, $i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \begin{bmatrix} \nabla h_1(\mathbf{x}^*) & \ldots & \nabla h_m(\mathbf{x}^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0$, $i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \ge 0$ are the Kuhn-Tucker multipliers.

9. Penalty & Barrier Functions

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \leq 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$

Q1 (a) (i) Poles at $z = n\pi$ with residues $n\pi(n\pi - 1)^{1/2}(-1)^n$

Branch point at $z = 1$

(ii) Pole of order 3 at $z = -1$ with residue $\frac{1}{4\sqrt{2}}$

Branch points at $z = \pm i$

(b) $\frac{\pi}{\sqrt{3}}$

Q2 (a) About $z = 2$: $f(z) = \frac{1}{4}\frac{\cos 2}{z-2} - \frac{1}{4}(\cos 2 + \sin 2)$

About $z = 0$: $f(z) = -\frac{1}{2z^2} - \frac{1}{4z} + \frac{1}{4}$

(b) $F(\omega) = \frac{1}{a - i\omega}$

Q3 (a) Minimize $f(\underline{x}) = -0.30x_1 - 0.34x_2 - 0.37x_3 - 0.39x_4$

subject to $x_1 + 0.9x_2 + 0.8x_3 + 0.7x_4 = 64\times10^3$

and $0.1x_2 + 0.2x_3 + 0.3x_4 = 12\times10^3$

(b) $\underline{x} = (0, 32\times10^3, 44\times10^3, 0)$

(c) -

Q4 (a) Minimise $f = -I = \frac{-31.5r^2n^2}{6r + 9l + 10t}$

subject to $2\pi rlt - V = 0$

and $2\pi rn - L = 0$

(b) $l = 1/27$ m, $r = 1/18$ m, $t = 1/30$ m, $n = 18$ turns, $I = 31.5$ μH

(c) −0.105 μH, 0.63 μH