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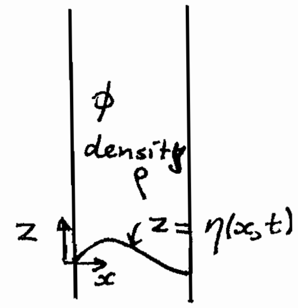
Module 4A10 FLOW INSTABILITY 2006

Solutions

Qu 1



For $\phi(x,z,t)$ to be a suitable velocity potential, we must have

(i) $\nabla^2\phi = 0$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = Ae^{st}\cos\left(\frac{n\pi x}{d}\right)e^{-n\pi z/d} \times \left(-\frac{n^2\pi^2}{d^2} + \frac{n^2\pi^2}{d^2}\right) = 0 \checkmark$$

(ii) $\phi \longrightarrow 0$ as $z \longrightarrow \infty$, $\checkmark$

(iii) $\frac{\partial\phi}{\partial x} = 0$ on rigid walls $x = 0, d$

$$\frac{\partial\phi}{\partial x} = -Ae^{st}\frac{n\pi}{d}\sin\left(\frac{n\pi x}{d}\right)$$

$= 0$ on $x=0$, $= 0$ on $x=d$ for n integer $\checkmark$

Hence $\underline{\phi(x,z,t) = Ae^{st}\cos\left(\frac{n\pi x}{d}\right)e^{-n\pi z/d}}$ is a suitable velocity potential. [25%]

The kinematic boundary condition is $\frac{\partial\eta}{\partial t} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0}$.

Hence $\eta_0 s = A\left(-\frac{n\pi}{d}\right)$ $\quad \underline{\eta_0 = -\frac{n\pi}{ds}A}$ (i)

The value of s follows from the pressure boundary condition.

$$p(x,\eta,t) - p_a = \sigma\frac{\partial^2\eta}{\partial x^2}$$

where $p(x,\eta,t)$ denotes the pressure in the liquid at $z=\eta$

$$p(x,\eta,t) = -\rho\frac{\partial\phi}{\partial t} - \rho g\eta$$

Hence $-\rho sA = \eta_0\left(-\sigma\left(\frac{n\pi}{d}\right)^2 + \rho g\right)$ $\quad \underline{A = -\frac{1}{s}\eta_0\left(g - \frac{\sigma}{\rho}\frac{n^2\pi^2}{d^2}\right)}$ [another relationship between η_0 and A]

Substituting for η_0 from (i) $\quad A = +\frac{n\pi}{ds^2}A\left(g - \frac{\sigma}{\rho}\frac{n^2\pi^2}{d^2}\right)$

$$\underline{s^2 = \frac{n\pi}{d}\left(g - \frac{\sigma}{\rho}\frac{n^2\pi^2}{d^2}\right)}$$

[40%]

If $\frac{\sigma}{\rho}\frac{\pi^2}{d^2} \geqslant g$, Real $s = 0$ for all possible modes n, and the surface is stable

i.e. $\underline{\text{stability for } \sigma \geqslant \rho g d^2/\pi^2}$

Qu 1 cont.)

If σ is reduced to just below $\sigma = \rho g d^2/\pi^2$

the $n=1$ mode will have Real $s>0$ and grow while $n \geq 2$ modes will still be stable. The resulting flow therefore has

$$\phi(x,z,t) = A e^{st} \cos\left(\frac{\pi x}{d}\right) e^{-\pi z/d}, \quad \eta(x,t) = \eta_0 e^{st} \cos\left(\frac{\pi x}{d}\right)$$

where $s = \left[\frac{\pi}{d} \cdot \left(g - \frac{\sigma \pi^2}{d^2}\right)\right]^{1/2}$ [20%]

The $\cos(\pi x/d)$ means that the surface displacement has the form

(It is of cource arbitrary and depends on the sign of η_0 whether the downward moving portion is in $0<x<d/2$ or $d/2<x<d$.) [15%]

Qu 2 (I) a boundary layer flow over a curved wall.

a) The two mechanisms involve viscosity and curvature of streamlines [5%]

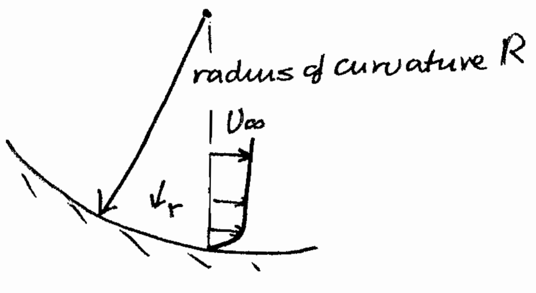
b) Viscosity i) Boundary layer flows are inviscidly stable by Rayleigh's inflexion point theorem ($U'' \neq 0$), but viscosity is a destabilising mechanism. It has the effect of changing the phase relationship between pressure and velocity in linear disturbances and enabling a transfer of energy from the mean flow into perturbations which then grow. [5%]

ii) Viscosity can stabilise the flow at very low Reynolds numbers [5%]

iii) The relevant nondimensional parameter is the Reynolds number Re_δ based on the local boundary layer thickness $Re_\delta = \frac{U_\infty \delta(x)}{\nu}$. [5%]

(iv) $\delta(x)$ increases from the leading edge of the plate, proportional to $x^{1/2}$, so Re_δ increases along the plate. The disturbances are Tollmien-Schlichting waves which travel downstream. At low Re_δ (ie near the leading edge) all waves decay as they propagate. At a particular axial position a wave of a particular frequency ω starts to grow but it propagates downstream to larger Re_δ where it decays. When one of the waves grows to exceed a critical value (and the Re_δ at which this occurs depends on the amplitude of the oncoming flow disturbances) nonlinear and 3D effects become important leading ultimately to turbulence. [5%]

Curvature i) If the surface is concave there is a second mechanism for instability because then rV decreases with increasing r. By Rayleigh's criterion for flows with curved streamlines this is inviscidly unstable.



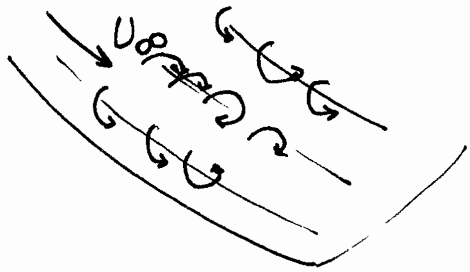
The physical mechanism is that a pressure gradient $\frac{\partial p}{\partial r} = \rho \frac{V^2}{r}$ is required to balance the radial acceleration. A fluid particle displaced to larger r has high velocity than the surrounding particles and hence the local pressure gradient is not sufficient to restore its position and the particle continues to move to larger radius leading to instability [5%]

(ii) viscosity tends to stabilise the flow [5%]

(iii) the relevant nondimensional parameter is $\frac{U_\infty \delta}{\nu}\left(\frac{\delta}{R}\right)^{1/2}$. [5%]

Qu 2 cont.)

iv) When the flow is unstable Görtler vortices are formed. These vortices have their axes in the direction of the free stream. They occur as pairs of counter-rotating vortices, this cell pattern repeating across the span.



[5%]

c) It is radius of curvature that determines which instability happens on a concave surface, the relevant non-dimensional parameter being $U_\infty R/\nu$. Also expect an effect from oncoming flow disturbances or surface roughness which would both tend to promote the onset of turbulence. [5%]

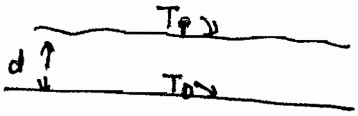
II) <u>Fluid heated from below</u>

a) The two mechanisms are buoyancy and the variation of surface tension with temperature. [5%]

b) <u>buoyancy</u> (i) heating from below leads to instability because as a particle of fluid rises it is hotter and hence less dense than its surroundings, and continues to rise. [5%]

(ii) stabilising influences are viscosity and the diffusion of heat. [5%]

(iii) For layer of liquid with heated bottom surface T_0, upper surface T_1, depth d, the relevant nondimensional parameter is the Rayleigh number



$$Ra = \frac{g\alpha(T_0 - T_1)d^3}{\nu\kappa},$$

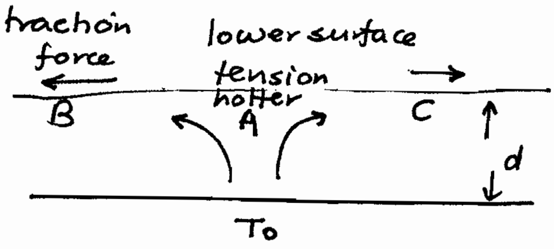
where ν is the kinematic viscosity, κ the thermal diffusivity ($=\lambda/\rho c_p$, λ thermal conductivity), g acceleration due to gravity, α coefficient of thermal expansion. [5%]

(iv) After the onset of instability, steady convection rolls form, hot liquid rises, is cooled at the surface and then falls. The rolls are cylindrical, with the axes of the rolls, depending on the container: parallel to the shortest side for a rectangular container, or circles for a cylindrical container.

convection rolls

[5%]

Qu 2 cont.)



Variation of surface tension

i) mechanism, upwelling fluid is hotter when it meeting the surface. It therefore has less surface tension than the fluid at B and C. Surface traction therefore pulls the fluid away from A, so amplifying the motion. [5%]

ii) Viscosity and thermal diffusion are again stablising influences [5%]

iii) The relevant nondimensional number is the Maragoni number

$$Ma = -\frac{d\sigma}{dT} \frac{(T_0 - T_1)d}{\rho \nu \kappa} \qquad \sigma = \text{surface tension.}$$ [5%]

iv) The fluid breaks up into steady hexagonal cells, where the upwelling liquid rises in the centre of the hexagonal and sinks around the edges. [5%]

c) The ratio of the Maragoni and Rayleigh numbers determines which form of instability happens:

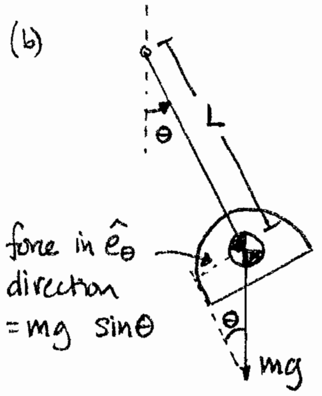
$$\frac{Ma}{Ra} = -\frac{d\sigma}{dT} \frac{1}{g\alpha d^2}$$

ie The surface tension-driven instability is more likely for thin layers. [5%]

Q 3 (a) The Reynolds number based on the diameter of the lamp is 20,000 so the flow will separate at the edges of the lamp creating two parallel shear layers. Initially we would expect vortex shedding at a Strouhal number of approximately 0.2, which corresponds to 0.67 Hz. This will start the lamp moving from side to side and set up the main motion, which is gallop.

When the lamp moves across the flow, the apparent angle of attack is such that the flow induces a small force in the direction of motion. This force is proportional to the lamp's velocity $L\dot{\theta}$. This force acts whenever the lamp moves across the flow. A pendulum-like motion follows, with energy put into the motion via the mechanism just described. [20%]

(b)



The moment due to gravity acting on the lamp is in the opposite direction to the angular displacement, θ. The force balance on the left, performed per unit depth into the page, shows that the magnitude is $mgL\sin\theta$.

Therefore the moment about the ceiling pivot is $-mgL\sin\theta$. [5%]

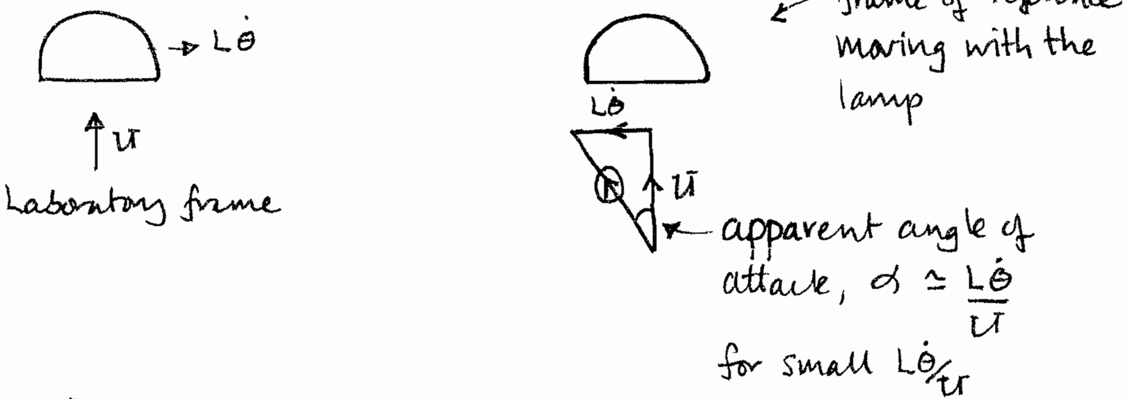
This moment balances with the moment of inertia, L^2m, multiplied by the angular acceleration: $L^2 m\ddot{\theta} = -mgL\sin\theta$

$$\Rightarrow \quad L\ddot{\theta} + g\sin\theta = 0$$

If θ is small then $\sin\theta \simeq \theta$ [5%]

$$\Rightarrow \quad L\ddot{\theta} + g\theta = 0$$

(c) When the radiator is on, the lamp sits in a uniform vertical flow. Let us imagine a small oscillation of the lamp such that the horizontal velocity is $\simeq L\dot{\theta}$.



From Fig. 2b it can be seen that the flow, at this apparent angle of attack, gives rise to a force in the direction of motion: $F = C_F(\alpha)\left[\frac{1}{2}\rho U^2 D\right]$ (per unit depth into page)

(cont.

Q3 cont.)

If α is small then $C_F \simeq \frac{dC_F}{d\alpha}\alpha$ so the force per unit depth is approximately equal to: $F \simeq \frac{dC_F}{d\alpha}\left[\frac{1}{2}\rho U^2 D\right]\alpha \simeq \frac{dC_F}{d\alpha}\left[\frac{1}{2}\rho U^2 D\right]\frac{L\dot{\theta}}{U}$ [20%]

This is in the direction of motion so it is positive. It must be multiplied by L to create a moment. The equation of motion becomes:

$$L^2 m\ddot{\theta} + \gamma L\dot{\theta} + mgL\theta = \left[\frac{1}{2}\rho U^2 D\right]\frac{dC_F}{d\alpha}\frac{L^2\dot{\theta}}{U}$$ (expressed in terms of moments)

and in terms consistent with the question this is:

$$L\ddot{\theta} + \frac{\gamma}{m}\dot{\theta} + g\theta = \left[\frac{1}{2}\frac{\rho U D L}{m}\right]\frac{dC_F}{d\alpha}\dot{\theta}$$ [10%]

The forcing term can now be combined with the damping term, $\frac{\gamma}{m}$, because both are proportional to $\dot{\theta}$:

$$L\ddot{\theta} + \left\{\frac{\gamma}{m} - \frac{1}{2}\frac{\rho U D L}{m}\frac{dC_F}{d\alpha}\right\}\dot{\theta} + g\theta = 0$$ [10%]

Oscillations will grow when the net damping term is negative, which is when:

$$\frac{1}{2}\frac{\rho U D L}{m}\frac{dC_F}{d\alpha} > \frac{\gamma}{m}$$

$\frac{dC_F}{d\alpha}$ at $\alpha = 0$ from Fig. 2b

$$\Rightarrow \gamma < \frac{1}{2}\rho U D L\frac{dC_F}{d\alpha} = \frac{1}{2} * 1.2 * 1 * 0.3 * 2.45 * 1$$

$$\Rightarrow \gamma < 0.441 \text{ kg s}^{-2}$$ [10%]

The frequency of these oscillations is the natural frequency of the undamped system, since the net damping term is zero (when they start). By substituting $\theta = \theta_0 \sin(\omega t)$ into $L\ddot{\theta} + g\theta = 0$ one obtains $\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{2.45}}$
$= 2 \text{ rad s}^{-1}$ [10%]

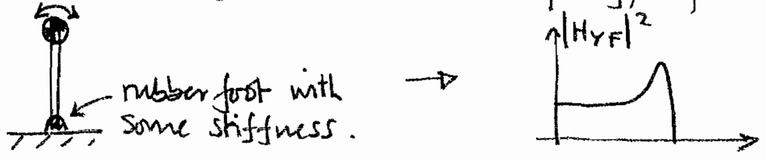
(d) In order to prevent this motion, the front of the lamp could be made the same shape as the back so that $dC_F/d\alpha = 0$. Alternatively, the whole lamp could be made into an aerofoil shape so that $dC_F/d\alpha$ becomes negative. Any shape that avoids positive $dC_F/d\alpha$ will avoid gallop due to soft excitation. [10%]

4. (a) $$S_{FF} = |H_{UF}|^2 S_{uu}$$

S_{FF} is the power spectral density of the aerodynamic forcing.
S_{uu} " " " " " " the velocity fluctuations in the wind.
$|H_{UF}|^2$ is the aerodynamic admittance.

S_{FF} and S_{uu} represent the contributions to the mean square forcing (F) or velocity fluctuation (u) as a function of frequency, ω. The aerodynamic admittance relates the velocity fluctuation to the force fluctuation as a function of frequency. The aerodynamic admittance would have to be obtained from experimental measurements on a wind tunnel model or from estimated from previous measurements on similar masts. S_{uu} can be measured experimentally in Cambridge and multiplied by $|H_{UF}|^2$ to obtain S_{FF}. [10%]

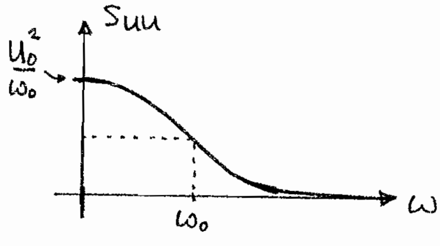
The mechanical admittance $|H_{YF}|^2$ is required to evaluate the response of the structure to the forcing. This could be estimated from a simple model of a mass on a spring, representing the rubber feet:



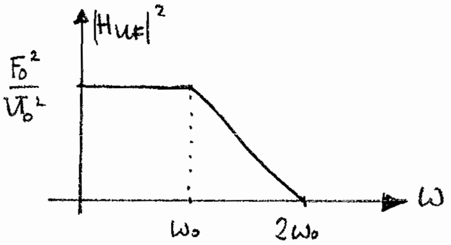
The power spectral density of the response is then given by:

$$S_{YY} = |H_{YF}|^2 S_{FF} = |H_{YF}|^2 |H_{UF}|^2 S_{uu}$$ [10%]

(b) $$S_{uu} = \frac{U_0^2/\omega_0}{1 + \omega^2/\omega_0^2}$$:



$|H_{UF}|^2 =$ see question :



$$S_{FF} = |H_{UF}|^2 S_{uu} \quad \text{and} \quad \overline{F'^2} = \int_0^\infty S_{FF}\, d\omega = \int_0^\infty |H_{UF}|^2 S_{uu}\, d\omega$$

Substitute for $|H_{UF}|^2$ and S_{uu} and integrate. [20%]

(cont.

Q4 cont.)

$$\overline{F'^2} = \int_0^{\omega_0} \frac{F_0^2}{U_0^2}\left(\frac{U_0^2/\omega_0}{1+\omega^2/\omega_0^2}\right) d\omega + \int_{\omega_0}^{2\omega_0} \frac{F_0^2}{U_0^2}\left(2-\frac{\omega}{\omega_0}\right)\left(\frac{U_0^2/\omega_0}{1+\omega^2/\omega_0^2}\right) d\omega$$

$$= \frac{F_0^2}{\omega_0}\left\{ \int_0^{\omega_0} \frac{1}{1+\omega^2/\omega_0^2}\, d\omega + 2\int_{\omega_0}^{2\omega_0} \frac{1}{1+\omega^2/\omega_0^2}\, d\omega - \int_{\omega_0}^{2\omega_0} \frac{\omega/\omega_0}{1+\omega^2/\omega_0^2}\, d\omega \right\}$$

Change variable : $x = \omega/\omega_0 \Rightarrow \omega_0\, dx = d\omega$

$$\Rightarrow \overline{F'^2} = F_0^2 \left\{ \int_0^1 \frac{1}{1+x^2}\, dx + 2\int_1^2 \frac{1}{1+x^2}\, dx - \int_1^2 \frac{x}{1+x^2}\, dx \right\}$$

$$= F_0^2 \left\{ \left[\tan^{-1} x\right]_0^1 + \left[2\tan^{-1} x\right]_1^2 - \left[\tfrac{1}{2}\ln(1+x^2)\right]_1^2 \right\}$$

$$= 0.97\, F_0^2$$

[40%]

(c) In a steady wind there will be vortex shedding and possibly gallop on the masts' structures. However, the most significant unsteady force will arise from vortex shedding from the cylindrical aerial. In any significant wind the Reynolds number will be large (greater than 10^3) so the Strouhal number will be around 0.2 :

$$St = \frac{fD}{U} \simeq 0.2$$

This will give rise to transverse unsteady forces on the structure.

[20%]

End of paper

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Matthew Juniper

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