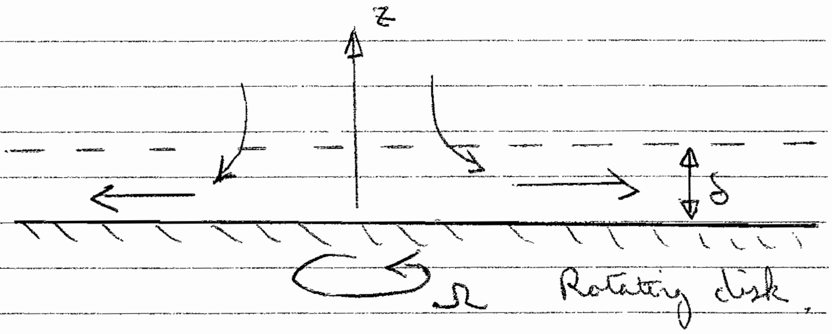
4A12 Turbulence 2005-06

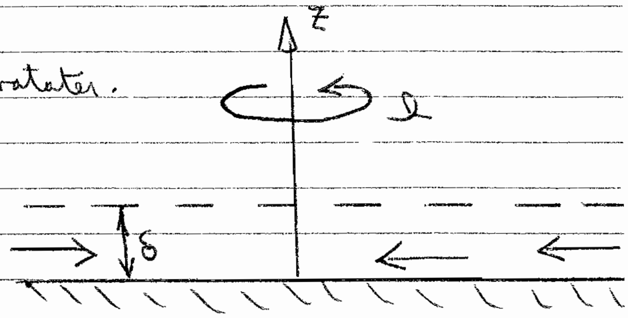
1 a) Kármán Layer



Fluid in layer rotates with disk due to no-slip condition.

Nothing to balance centrifugal force so fluid spirs outward.

Bödewadt layer



This time disk is stationary & fluid rotates. Outside layer rotation sets up a radial pressure gradient

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r$$

This pressure gradient is imposed on boundary layer where u_θ is smaller

The excess pressure gradient drives a radial inflow in boundary layer

b) In either layer there is radial force:

Kármán layer it is $\rho \frac{u_\theta^2}{r}$ (centrifugal force)

Bödewadt layer it is $\rho \frac{u_\theta^2}{r} - \underbrace{\rho \frac{(\Omega r)^2}{r}}_{\partial p/\partial r} = -\rho \frac{\Omega^2 r^2 - u_\theta^2}{r}$

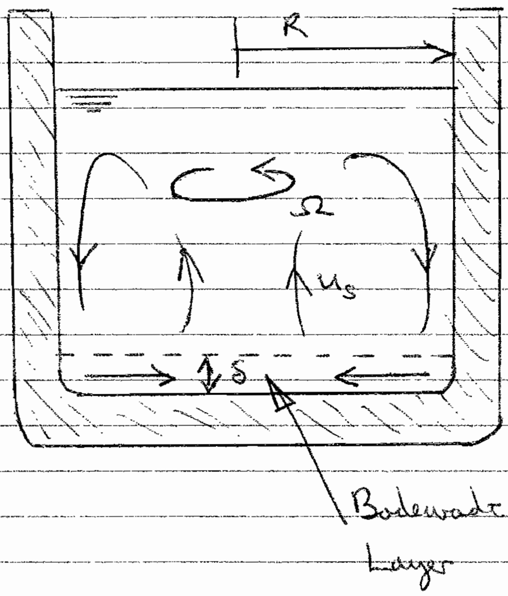
In the steady state this is balanced by viscous stresses. The radial viscous force is $\rho \nu \, \partial^2 u_r / \partial z^2 \sim \rho \nu \, u_r / \delta^2$

Thus,

$$\rho \frac{u_\theta^2}{r} \sim \rho \nu \frac{u_r}{\delta^2}$$

If $u_r \sim u_\theta$, $\quad \frac{\nu}{\delta^2} \sim \frac{u_\theta}{r} \sim \Omega \Rightarrow \delta \sim \sqrt{\nu/\Omega}$

c)



Fluid comes to rest after contents of tea have been flushed through the Bodewadt layer where the KE of fluid is destroyed.

Spin down time is therefor turn-over time of secondary flow.

In Bodewadt layer $[u_r]_\delta \sim u_\theta \sim \Omega R$

Continuity gives magnitude of secondary flow in core as

$$u_s \times R \sim [u_r]_\delta \times \delta \sim (\Omega R)\,\delta$$

But $\delta \sim \sqrt{\nu/\Omega}$

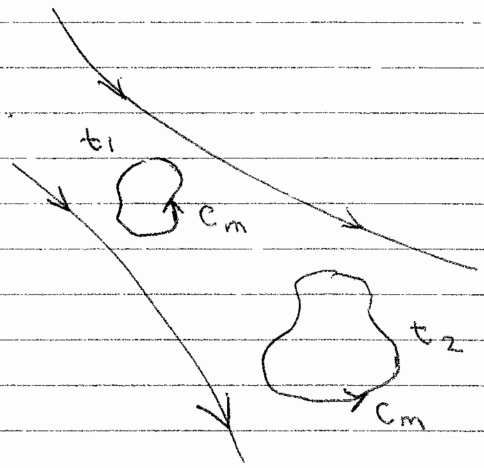
$$\Rightarrow \quad u_s \sim \Omega\delta \sim \sqrt{\nu\Omega}$$

Thus turn-over time of secondary flow is,

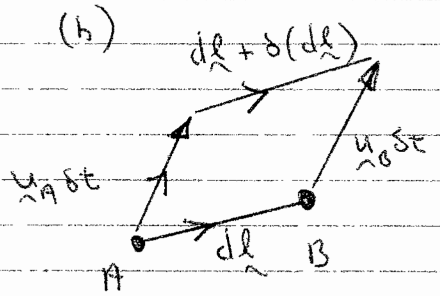
$$\tau = \frac{R}{u_s} \sim \frac{R}{\sqrt{\nu\Omega}}$$

② (a) <u>Kelvin's theorem</u>: $\Gamma = \oint_{C_m} \underset{\sim}{u} \cdot d\underset{\sim}{l} = \text{constant}$ if $\nu = 0$

for a material curve (dye line) C_m.



(b)



In time δt,

$$\delta(d\underset{\sim}{l}) = \underset{\sim}{u}_B \delta t - \underset{\sim}{u}_A \delta t$$

$$= (\underset{\sim}{u}_B - \underset{\sim}{u}_A)\,\delta t$$

$$\Rightarrow \quad \frac{D}{Dt}(d\underset{\sim}{l}) = \underset{\sim}{u}_B - \underset{\sim}{u}_A = (d\underset{\sim}{l}\cdot\nabla)\,\underset{\sim}{u}_A$$

c.f. $(\underset{\sim}{u}_B)_x - (\underset{\sim}{u}_A)_x = (d\underset{\sim}{l}\cdot\nabla)\,u_x$, similarly for y, z.

Thus $$\underline{\underline{\frac{D}{Dt}\, d\underset{\sim}{l} = (d\underset{\sim}{l}\cdot\nabla)\,\underset{\sim}{u}}}$$

Let $d\underset{\sim}{l} = \alpha\,\underset{\sim}{\omega}$

$$\frac{D}{Dt}(\alpha\underset{\sim}{\omega}) = (\alpha\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u}$$

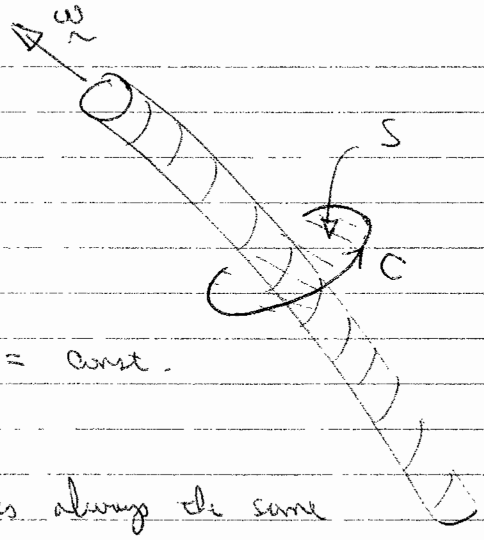
$$\Rightarrow \quad \underset{\sim}{\omega}\frac{D\alpha}{Dt} + \alpha\frac{D\underset{\sim}{\omega}}{Dt} = \alpha(\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u}$$

But $\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\,\underset{\sim}{u}$ in an inviscid fluid,

$$\Rightarrow \quad \underline{\underline{\frac{D\alpha}{Dt} = 0}}$$

This is <u>Helmholtz' law.</u> ③

(c)



From Kelvin's theorem

$$\oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l} = \int_S \underset{\sim}{\omega} \cdot d\underset{\sim}{S} = \text{const.}$$

Thus flux through C is always the same

Thus C always encircles flux tube.

This is true for any material curve that encloses the vortex tube.

Since these curves all move with the fluid, so must the vortex tube.

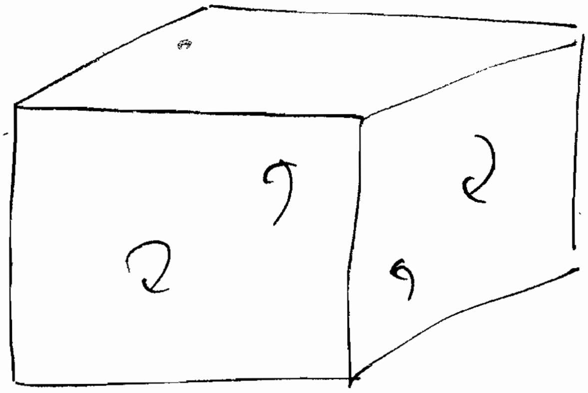
3 (a) Turbulence is produced close to the floor of the room by the motion of the people. In the TKE equation, the fluctuating force terms are responsible for turbulence generation : $\overline{f_1' u_1'} + \overline{f_2' u_2'}$ in 1 & 2 directions only (horizontal)

There is no mean air motion $\Rightarrow \frac{\partial \bar{u}_i}{\partial x_j} = 0 \Rightarrow$ no production

Hence turbulence is produced at low z & diffuses to larger z (above the guests' head) by the action of the transport terms in the TKE.

At low z, turbulence is probably ~~highly~~ anisotropic, since there is no generation in the z-direction. Eventually, as we go upwards, the pressure re-distribution term makes all 3 components approximately equal.

(b)



Isotropic ~~stationary~~ turbulence

homogeneous in all directions

$\Rightarrow \frac{d}{dt}\left(\frac{3}{2}u^2\right) = $ production − dissipation

$V = 20 \times 20 \times 8 \ m^3$

$\rho =$ air density at room $T = 1.2 \ kg/m^3$

Power in = Dissipation (for steady state)

$\Leftrightarrow$ (# of people) $\times$ (power per person) $= \varepsilon \times (\rho V)$

$$\Leftrightarrow \quad \varepsilon = \frac{400 \cdot 1}{1.2 \times 20 \times 20 \times 8} = 0.1 \quad m^2/s^3$$

If $u \sim 0.5$ m/s, using $\varepsilon = \frac{u^3}{L} \Rightarrow L = 1.25$ m

(Some students failed to notice that u was given and estimated L as the "lengthscale of the turbulence production mechanism", ie. the distance between guests. From ε, this gave u. Not a crazy approach...)

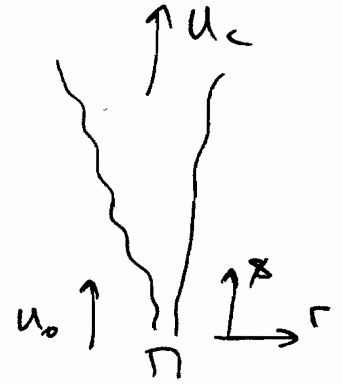
Turbulence timescale : $\frac{L}{u} \sim 2.5$ s

(c) $\eta_K = L \cdot (Re_t)^{-3/4}$

$$= L \cdot \left(\frac{uL}{\nu}\right)^{-3/4} = 0.4 \ mm$$

Hence 20 mm balloon falls in the inertial range $\Rightarrow$ cannot follow all turbulent motions.

④



(a) Self-similarity means $\bar{U}(x,r) = U_c(x) \cdot f(\eta)$, $\eta = \frac{r}{\delta}$

and $\delta = \delta(x)$

Momentum flow rate is conserved

$\Rightarrow$ $\dot{M} = \int_0^\infty \rho\, 2\pi r\, \bar{U}^2(r,x)\, dr$ is independent of x

$\Rightarrow$ $\int_0^\infty \rho\, 2\pi (\delta\eta)\, U_c^2 f^2(\eta)\, \delta\, d\eta$ is not a f of x

$= \rho\, 2\pi\, \delta^2 U_c^2 \int_0^\infty \eta f^2(\eta)\, d\eta$ is not a f of x

$\Rightarrow$ since $\delta \sim x$, $U_c \sim x^{-1}$.

(b) If Reynolds stresses are self-similar,

$\overline{u_i u_j} = Q_{ij}\, g(\eta)$ & Q_{ij} = f of x only

$\Rightarrow$ characteristic velocity $u \sim u(x)\, g(\eta)$

$\Rightarrow$ $\frac{u}{\bar{u}} = \text{constant}(\eta) \cdot \frac{u(x)}{U_c(x)}$ $\sim$ constant f of η only

since $u/U_c \sim$ constant for self-preserving jet

Eddy viscosity $\sim C\, u(x)\, \delta(x)$

$\sim$ Constant with x since $u(x) \sim x^{-1}$ (since $U_c \sim x^{-1}$)

(c) mass flow rate =

$$\dot{m}(x) = \int_0^\infty \rho 2\pi r \, \bar{u}(x,r) \, dr$$

$$= \rho 2\pi \underbrace{\delta^2(x)}_{\delta \sim x} \underbrace{u_c(x)}_{\sim x^{-1}} \underbrace{\int_0^\infty \eta f(\eta) \, d\eta}_{\text{constant with } x}$$

$$\Rightarrow \quad \dot{m}(x) \sim x$$