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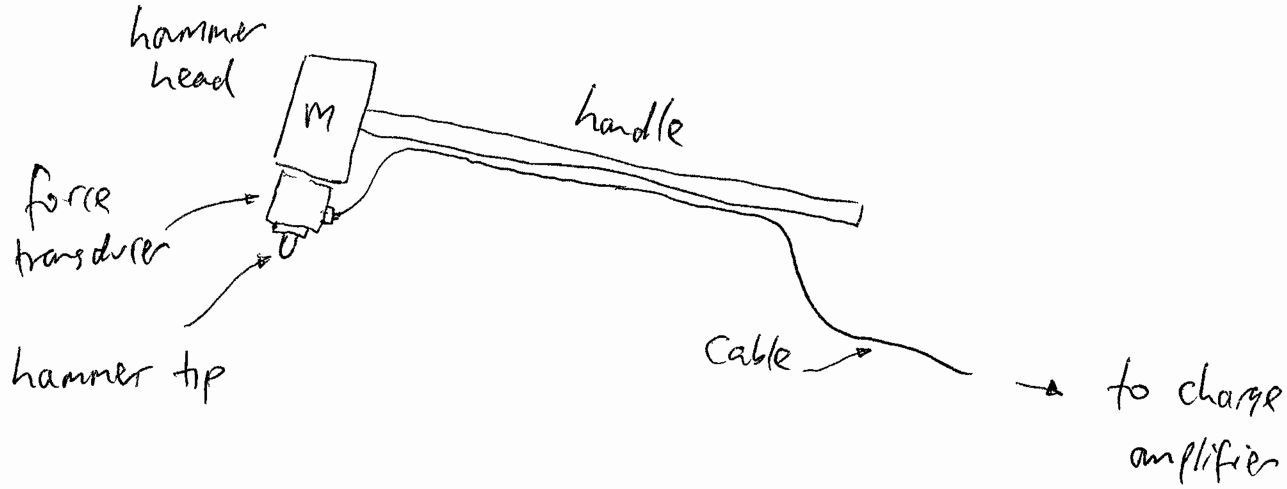
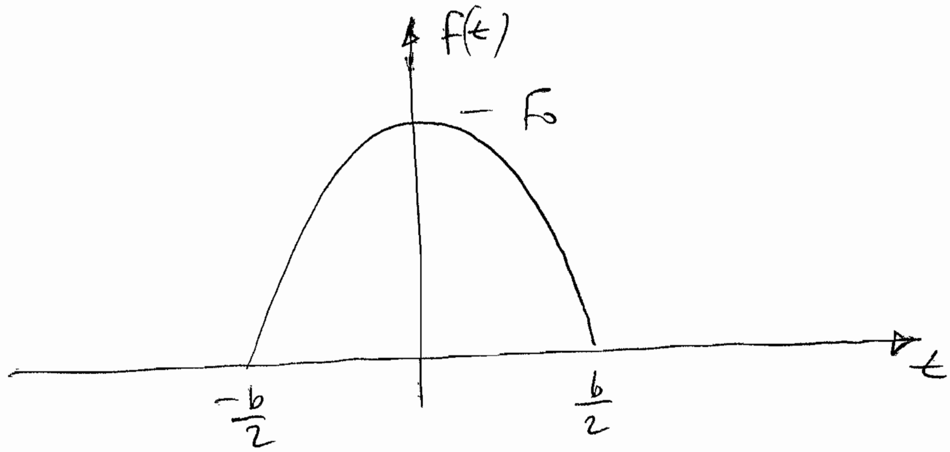
Part IIB 4C6 Solutions 2006

1 (a)



(b) (i)



$$\Omega \frac{b}{2} = \frac{\pi}{2} \quad \therefore \quad \underline{\underline{\Omega = \frac{\pi}{b}}}$$

$$\Omega = \sqrt{\frac{k}{m}}$$

"natural frequency" of cosine pulse

(ii)
$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{F_0}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos \Omega t \; e^{-i\omega t}\, dt$$

(This is the "long" way)

$$= \frac{F_0}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cancel{\cos\Omega t}\,\frac{1}{2}\left(e^{i\Omega t}+e^{-i\Omega t}\right)e^{-i\omega t}\,dt$$

this is the long way

$$= \frac{F_0}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(e^{i(\Omega-\omega)t}+e^{-i(\Omega+\omega)t}\right)dt$$

$$= \frac{F_0}{4\pi}\left[\frac{e^{i(\Omega-\omega)t}}{i(\Omega-\omega)} - \frac{e^{-i(\Omega+\omega)t}}{i(\Omega+\omega)}\right]_{\cancel{-\frac{b}{2}}\ -\frac{\pi}{2\Omega}}^{\cancel{\frac{b}{2}}\ \frac{\pi}{2\Omega}}$$

$$= \frac{F_0}{4\pi}\left[\frac{2\sin(\Omega-\omega)\frac{\pi}{2\Omega}}{\cancel{i}(\Omega-\omega)} + \frac{2\sin(\Omega+\omega)\frac{\pi}{2\Omega}}{\Omega+\omega}\right]$$

$$= \frac{F_0}{2\pi}\left[\frac{\sin\left(\frac{\pi}{2}-\frac{\pi\omega}{2\Omega}\right)}{\Omega-\omega} + \frac{\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\frac{\omega}{\Omega}\right)}{\Omega+\omega}\right]$$

but $\sin\left(\frac{\pi}{2}-x\right) = \cos x$

& $\sin\left(\frac{\pi}{2}+x\right) = \cos x$

$$\therefore\ F(\omega) = \frac{F_0}{2\pi}\left(\frac{\cos\frac{\pi\omega}{2\Omega}}{\Omega-\omega} + \frac{\cos\frac{\pi\omega}{2\Omega}}{\Omega+\omega}\right)$$

$$= \frac{F_0}{2\pi}\cos\frac{\pi\omega}{2\Omega}\left[\frac{\Omega+\omega+\Omega-\omega}{\Omega^2-\omega^2}\right]$$

$$= \frac{\Omega F_0}{\pi}\ \frac{\cos\frac{\pi\omega}{2\Omega}}{\Omega^2-\omega^2}$$

$$\therefore\quad B = \frac{\Omega F_0}{\pi}$$

(ii) Alternatively — and much quicker (preferred)

Consider $F(\omega)$, $\omega \to 0$ $= \dfrac{B}{\Omega^2}$

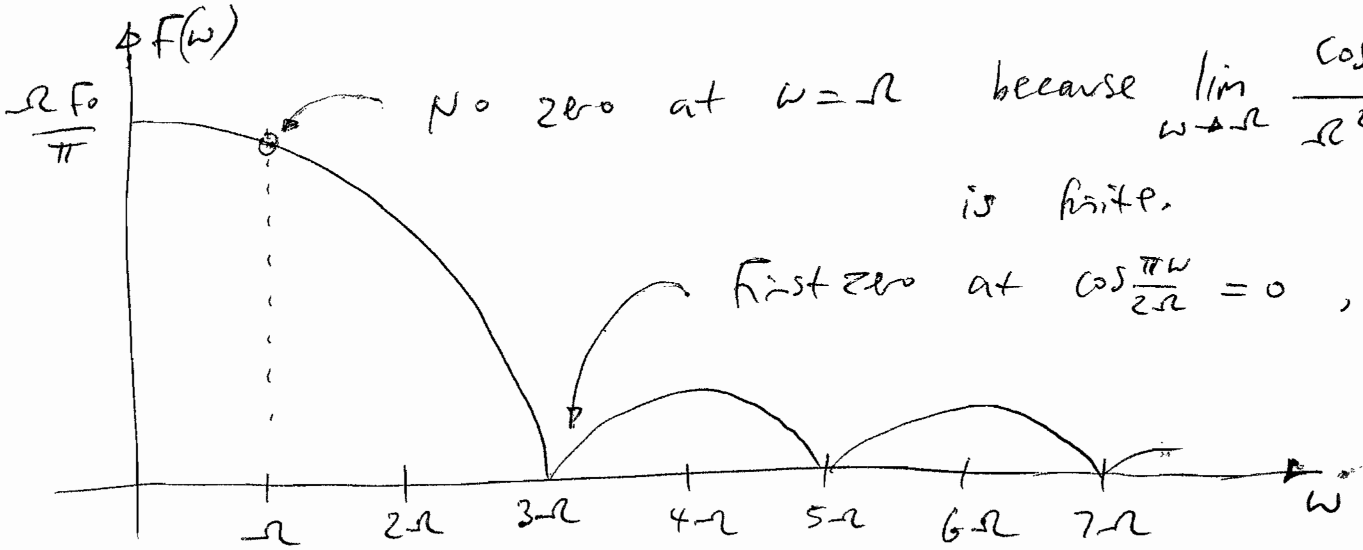
and this is the d.c. component of the impulse

$$\therefore \quad \frac{B}{\Omega^2} = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} F_0 \cos\Omega t \, dt$$

$$= \frac{1}{2\pi}\frac{F_0}{\Omega}\Big[\sin\Omega t\Big]_{-\frac{\pi}{2\Omega}}^{\frac{\pi}{2\Omega}}$$

$$= \frac{F_0}{\pi\Omega}$$

$$\therefore \quad B = \frac{\Omega F_0}{\pi}$$



No zero at $\omega = \Omega$ because $\lim\limits_{\omega \to \Omega} \dfrac{\cos\frac{\pi\omega}{2\Omega}}{\Omega^2 - \omega^2}$ is finite.

First zero at $\cos\frac{\pi\omega}{2\Omega} = 0$, $\frac{\pi\omega}{2\Omega} = \frac{3\pi}{2}$

Energy of impulse is here, $\omega < 3\Omega$

$$\therefore \quad 2\pi f < 3\frac{\pi}{b}$$

$$\therefore \quad f < \frac{1.5}{b}$$, roughly as stated.

2 (a) Material damping arises from dissipative effects of cyclic strain within the bulk of the material a structure is made of.

Metals: mainly from small movements of dislocations, so ductile metals tend to be more highly damped than hard metals or alloys with strongly pinned dislocations

Ceramics and glasses: provided they are uncracked they are built entirely with strong covalent bonds, so allow little atomic movement and have low damping

Polymers: Thermosoftening polymers can slip between chains held by Van Der Waals bonds, so have high damping. Cross-linked thermosetting polymers have less movement and so less damping

Boundary damping arises from locally nonlinear effects at joints or cracks. Processes include air pumping, micro-slipping and local impacts ("rattles or buzzes").

For built-up structures made in relatively low-damping materials, boundary damping effects usually dominate.

(b) (i) A tall building will usually be made of steelwork or reinforced concrete, infilled with panels of various possible materials. Steel and good-quality reinforced concrete both have relatively low damping. The damping of the building will come from the infill panels and, predominantly, from boundary damping effects from the joints and interior fittings of the building. To protect against vibrations from earthquakes, occasionally use a (large) tuned absorber at the top of the building to control "sway mode", but mainly look at detailed designs to maximise amount of boundary damping. Traditional buildings in earthquake areas often have timber/masonry "stripes" to allow "horizontal slip planes" to add high damping in shear

(ii) Blades have complicated composite construction: metal, GFRP, foam core would be typical. Bonding of components is high-quality for structural strength, so damping mainly from material effects (especially in GFRP + foam). Some boundary damping at root, where blade is held in a frictional socket.
Very hard to add damping without compromising aerodynamics or strength/fatigue life. Need to design to avoid blade resonances in operating frequency range.

(iii) Roof panel is steel, spot welded or seam welded to frame. Steel has relatively low damping. Boundary effects from welded edge will contribute moderate damping. But main damping from interior trim — soft panels on inside for cosmetic and safety purposes provide both material and boundary damping. To add even more damping, and help with rust prevention, a free-layer coating of bituminous material could be added to the steel, then covered by the trim panel.

3 (a) Potential energy $V = \frac{1}{2} k y^2 + \frac{1}{2} k z^2 + \frac{1}{2} S x^2$

$$\therefore \quad K = \begin{bmatrix} S & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

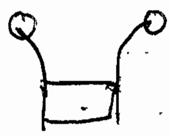
Kinetic energy $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (\dot{x} + \dot{y})^2 + \frac{1}{2} m (\dot{x} + \dot{z})^2$

$$\therefore \quad M = \begin{bmatrix} 3m & m & m \\ m & m & 0 \\ m & 0 & m \end{bmatrix}$$

(b) For approximate mode $[0, 1, -1]^t$ Rayleigh quotient gives

$$\omega^2 \simeq \frac{\frac{1}{2}k + \frac{1}{2}k}{\frac{1}{2}m + \frac{1}{2}m} = \frac{k}{m}$$

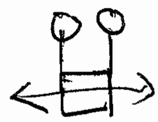
Mode is symmetric:



For $[1, 0, 0]$

$$\omega^2 \simeq \frac{\frac{1}{2} S}{\frac{1}{2} \cdot 3m} = \frac{S}{3m}$$

Mode is almost rigid-body:



For $[2, -3, -3]$

$$\omega^2 \simeq \frac{\frac{1}{2}k \cdot 9 + \frac{1}{2}k \cdot 9 + \frac{1}{2} S \cdot 4}{\frac{1}{2}m \cdot 4 + \frac{1}{2}m + \frac{1}{2}m} = \frac{9k + 2S}{3m}$$

Mode has base moving in opposite direction to the two tip masses.

(c) Damping turns $S \to S(1+i\eta)$. Can use the <u>same</u> expressions from Rayleigh's principle to give the approximate values for complex ω^2, provided the effect of damping is small enough that the mode shapes are still well approximated by the approximate undamped modes. $\eta \ll 1$ is enough to ensure this.

So from part (b): mode frequencies are

$$\omega^2 \simeq \frac{k}{m} \quad (\text{no damping})$$

$$\omega^2 \simeq \frac{S(1+i\eta)}{3m}$$

$$\omega^2 \simeq \frac{9k+2S(1+i\eta)}{3m}$$

Now $Q = \dfrac{Re(\omega)}{2 Im(\omega)}$, so if $\omega = a+ib$ with $b \ll a$, $\omega^2 \simeq a^2 + 2iab$.

$\therefore Q \simeq \dfrac{Re(\omega^2)}{Im(\omega^2)}$, and the three Q values are respectively ∞, $1/\eta$, $\dfrac{9k+2S}{2S\eta} \simeq \dfrac{9k}{2S\eta}$

(d) If η is not small, but still $S\eta \ll k$:
Undamped mode is still undamped.
Rigid-body mode will not obviously conform to the Rayleigh approximation, but since it is behaving as a single degree of freedom system, answer still correct.
Final mode still satisfies the approximation, since it only involves $S\eta$ compared to k.

"The" note of a tuning fork is the undamped symmetrical mode.

4 (a) "Blob of air" in neck of vessel behaves as a loose piston with a certain mass. It traps below it the air in the vessel, which compresses uniformly and produces a spring-like restoring force.

Equivalent system is simply

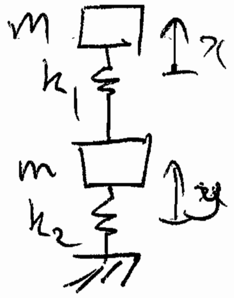
$m = LS\rho$ where L is effective length of neck, S is cross-sectional area and ρ is air density.

For thin-walled vessel $L = 2 \times 0.85a = 1.7a$ and $S = \pi a^2$

Now $\omega^2 = \dfrac{k}{m} = c^2 \dfrac{S}{VL}$, so $k = \dfrac{c^2 S}{VL} \cdot LS\rho$

So $m = 1.7\pi a^3 \rho$, $k = \rho c^2 \dfrac{(\pi a^2)^2}{V}$

(b) For coupled resonators, equivalent system is



where m is as above

$$k_1 = \frac{\rho c^2 \pi^2 a^4}{V_1}$$

$$k_2 = \frac{\rho c^2 \pi^2 a^4}{V_2}$$

In terms of displacements x, y as shown, mass matrix is $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, stiffness matrix is

$$K = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$$

So natural frequencies satisfy $|K - \omega^2 M| = 0$

$$\therefore (k_1 - \omega^2 m)(k_1 + k_2 - \omega^2 m) - k_1^2 = 0$$

$$\therefore k_1 k_2 - \omega^2 m (2k_1 + k_2) + \omega^4 m^2 = 0$$

$$\therefore \quad \omega^2 m = \tfrac{1}{2}\left[2k_1 + k_2 \pm \sqrt{(2k_1+k_2)^2 - 4k_1k_2}\,\right]$$
$$= \tfrac{1}{2}\left[2k_1 + k_2 \pm \sqrt{4k_1^2 + k_2^2}\,\right] \quad (1)$$

(c) Putting a plug in the lower hole is equivalent to fixing $y = 0$. This is one constraint, so the interlacing theorem will apply. New system has only the upper resonator, so it has the usual frequency governed by

$$\omega^2 = \frac{k_1}{m}, \quad \text{ie } \omega^2 m = k_1$$

So k_1 should lie between the two roots (1).

The + sign obviously gives an answer $> k_1$

With the − sign, $\omega^2 m = k_1 + \frac{k_2}{2} - \frac{\sqrt{4k_1^2 + k_2^2}}{2}$

But $\sqrt{4k_1^2+k_2^2} > k_2$ obviously, so $\omega^2 m < k_1$

So it does agree with the interlacing theorem.