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4C7 Random and Nonlinear Vibration 2006

1. a) To find the transfer function put $x(t) = x(\omega)e^{i\omega t}$, $y(t) = y(\omega)e^{i\omega t}$:

$$(-\omega^2 + 2i\beta\omega_n\omega + \omega_n^2)\, x(\omega) = \alpha\, y(\omega)$$

$$H(\omega) = \frac{x(\omega)}{y(\omega)} = \frac{\alpha}{\omega_n^2 - \omega^2 + 2i\beta\omega\omega_n}$$

The response spectrum is $S_{xx}(\omega) = |H(\omega)|^2 S_0$

$$\Rightarrow \quad \sigma_x^2 = \int_{-\infty}^{\infty} \frac{\alpha^2 S_0\, d\omega}{(\omega_n^2-\omega^2)^2 + (2\beta\omega\omega_n)^2} = \frac{\alpha^2 \pi S_0}{2\beta\omega_n^3}$$

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} \frac{\omega^2\alpha^2 S_0\, d\omega}{(\omega_n^2-\omega^2)^2 + (2\beta\omega\omega_n)^2} = \frac{\alpha^2 \pi S_0}{2\beta\omega_n}$$

standard integrals

The crossing rate: $\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2} = \frac{\omega_n}{2\pi}\, e^{-\frac{1}{2}(b/\sigma_x)^2}$

Failure probability $P = 1 - e^{-\nu_b^+ T} \Rightarrow \nu_b^+ T = -\ln(1-P)$

$$\Rightarrow \quad e^{-\frac{1}{2}(b/\sigma_x)^2} = -\frac{2\pi}{\omega_n T}\ln(1-P)$$

$$\Rightarrow \quad \left(\frac{b}{\sigma_x}\right)^2 = -2\ln\left[-\frac{2\pi}{\omega_n T}\ln(1-P)\right]$$

$$\Rightarrow \quad \frac{b^2\, 2\beta\omega_n^3}{\alpha^2 \pi S_0} = -2\ln\left[-\frac{2\pi}{\omega_n T}\ln(1-P)\right]$$

So, for failure probability less than P: $\underline{\beta > -\left(\frac{\alpha^2\pi S_0}{b^2\omega_n^3}\right)\ln\left[-\frac{2\pi}{\omega_n T}\ln(1-P)\right]}$ [40%]

b) For P small, $\ln(1-P) \simeq -P$. Number of cycles $N = \frac{T}{2\pi/\omega_n}$. Thus:

$$1 > -\left(\frac{\alpha^2\pi S_0}{\beta\omega_n^3 b^2}\right)\ln\left[\frac{P}{N}\right] \Rightarrow \underline{1 > -2\left(\frac{\sigma_x}{b}\right)^2 \ln\left[\frac{P}{N}\right]}$$ [30%]

c) Suppose $\beta = 0.05a$. Then σ_λ^2 proportional to $1/a$ and $(b/\sigma_\lambda)^2 = 16a$.

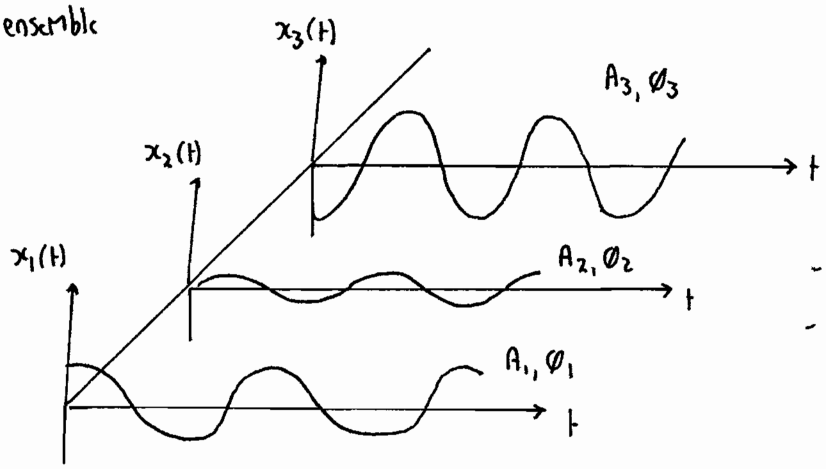
So we require $1 > -2\left(\frac{1}{16a}\right)\ln\left(\frac{P}{N}\right)$

Select $P = 10^{-6}$ for example $\Rightarrow$ $a > 2.67$ or $\underline{\beta > 0.134}$

or if $P = 10^{-7}$ then $a > 2.96$ or $\underline{\beta > 0.148}$

So a factor of 10 on P leads to a 10% change in β. [30%]

2 a) An ensemble



- Random phase and amplitude
- Frequency fixed

[20%]

b) $R_{xx}(\tau) = E[x(t)x(t+\tau)] = E[A^2 \sin(\omega t+\phi)\sin(\omega t+\omega\tau+\phi)]$

$= \frac{1}{2}E[A^2\cos(\omega\tau) - A^2\cos(2\omega t+\omega\tau+\phi)]$

Now $\underset{\substack{\uparrow \\ \text{any function}}}{E[F(A,\phi)]} = \iint F(A,\phi)\underset{\substack{\uparrow \\ \text{probability density function}}}{p(A,\phi)}\, dA\, d\phi$

A and ϕ statistically independent $\Rightarrow$ $p(A,\phi) = p(A)\underset{\substack{\uparrow \\ \text{given in question}}}{p(\phi)}$

$E[A^2\cos\omega\tau] = E[A^2]\cos\omega\tau = \alpha\cos\omega\tau$

$E[A^2\cos(2\omega t+\omega\tau+\phi)] = E[A^2]\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(2\omega t+\omega\tau+\phi)\, d\tau = 0$

$\Rightarrow$ $\underline{R_{xx}(\tau) = \frac{1}{2}\alpha\cos\omega\tau}$ [20%]

c) Stationary : statistal properties independent of time - yes

Ergodic : statistic across the ensemble the same as on any sample - no! [20%]

d) $M\ddot{z} + B\dot{z} + kz = A\sin(\omega t+\phi)$

Standard solution $z = \frac{A}{[(-\omega^2 M+k)^2 + (B\omega)^2]^{1/2}}\sin(\omega t+\phi+\psi)$

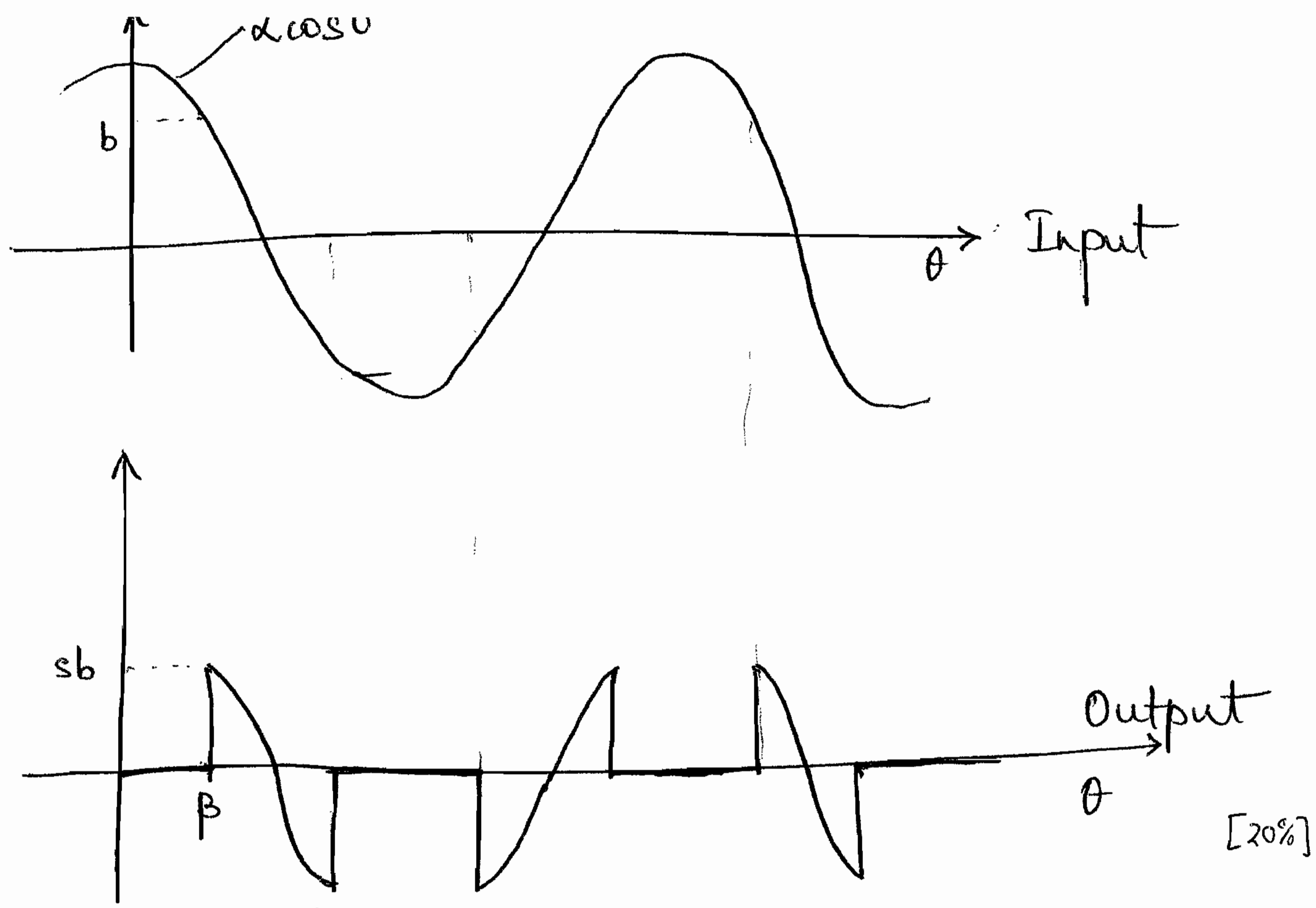
$\tan\psi = \frac{B\omega}{-\omega^2 M+k}$

By direct analogy with part b:

$$R_{zz}(\tau) = \left[\frac{\alpha/2}{(-\omega^2 M + k)^2 + (B\omega)^2}\right] \cos\omega\tau$$ [30%]

e) As in part (c), the process is not ergodic. [10%]

3(i)



[20%]

$\beta = \cos^{-1}(b/\alpha)$

(ii) The Describing Function (D) is the Fourier series coefficient of the fundamental frequency divided by input α.

$$D = \frac{1}{\alpha\pi}\int_0^{2\pi} (\text{output}) \cos\theta \, d\theta$$

$$= \frac{4}{\alpha\pi}\int_\beta^{\pi/2} s(\alpha\cos\theta)\cos\theta \, d\theta \quad ; \beta = \cos^{-1}(b/\alpha)$$

$$= \frac{2s}{\pi}\int_\beta^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{2s}{\pi}\left[\theta + \frac{\sin 2\theta}{2}\right]_\beta^{\pi/2}$$

$$D = \frac{2s}{\pi}\left[\pi/_2 - \beta - \frac{\sin 2\beta}{2}\right]$$

$$D = \frac{2s}{\pi}\left[\pi/_2 - \cos^{-1}(b/\alpha) - b/\alpha\sqrt{1 - b^2/\alpha^2}\right]$$

Limiting form: α very large

$b/\alpha \to 0$ and hence $\cos^{-1}(b/\alpha) \to \pi/2$

$$D \to \frac{2s}{\pi}\left[\pi/_2 - \pi/_2\right] \to 0$$

as we would expect for very large amplitudes α.

[60%]

(iii) The equation of motion using Describing function approximation:

$$m\ddot{x} + Dx \simeq \cancel{\alpha} f \cos\omega t$$

$x = \alpha \cos\omega t$ require

$$-m\omega^2\alpha + \frac{2s\alpha}{\pi}\left[\pi/_2 - \cos^{-1}(b/\alpha) - b/\alpha\sqrt{1 - b^2/\alpha^2}\right] \simeq a$$

This is a equation ~~for~~ relating α and ω.

[20%]

4 (i) Lagrangian $L = T - U$

$$= \frac{1}{2}m(l\sin\theta)^2\Omega^2 + \frac{1}{2}m(l\dot\theta)^2 - (-mgl\cos\theta)$$

$$L = \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta + \frac{1}{2}ml^2\sin^2\theta\,\Omega^2$$

Equation of motion:

$$\frac{d}{dt}\left(\frac{dL}{d\dot\theta}\right) - \frac{dL}{d\theta} = 0$$

$$\Rightarrow ml^2\ddot\theta - ml^2\Omega^2\sin\theta\cos\theta + mgl\sin\theta = 0.$$

$$\Rightarrow \ddot\theta + \frac{g}{l}\sin\theta - \Omega^2\frac{\sin 2\theta}{2} = 0.$$

[20%]

(ii) Equation of motion written as:—

$$\dot\theta = \phi$$

$$\dot\phi = -\frac{g}{l}\sin\theta + \Omega^2\frac{\sin 2\theta}{2}$$

Equilibrium or critical points occur when: $\dot\theta = \dot\phi = 0$

i.e. $\dot\theta = 0$ and $-g/l\,\sin\theta + \Omega^2\sin\theta\cos\theta = 0.$

(A) i.e. $\dot\theta = 0$, and $\sin\theta = 0 \Rightarrow \theta = 0, \pm\pi, \pm 2\pi, \ldots$

~~or~~ and

(B) $\dot\theta = 0$ and $\cos\theta = g/\Omega^2 l \Rightarrow \theta = \cos^{-1}(g/\Omega^2 l)$

Real roots of θ exist for case (B) when

$\Omega^2 > g/l$.

Effective potential function $V_{eff}(\theta)$ for a conservative system —

$$\frac{dV_{eff}}{d\theta} = g/l \sin\theta - \Omega^2 \frac{\sin 2\theta}{2}$$

$$\frac{d^2V_{eff}}{d\theta^2} = g/l \cos\theta - \Omega^2 \frac{\cos 2\theta}{\cancel{4}\cancel{2}} \times \cancel{2}$$

$\dot\theta = 0,\ \theta = 0$

$$\frac{d^2V_{eff}}{d\theta^2} = g/l - \Omega^2/\cancel{4}\quad \begin{aligned} &< 0 \text{ for } \Omega^2 > g/l \\ &> 0 \text{ for } \Omega^2 < g/l \end{aligned}$$

for stable equilibria $\frac{d^2V_{eff}}{d\theta^2} > 0$.

unstable " & $\frac{d^2V_{eff}}{d\theta^2} < 0$

$\therefore\ \dot\theta = 0,\ \theta = 0$ changes behaviour for a critical angular velocity $\Omega > \sqrt{g/l}$

$$\left.\frac{d^2V_{eff}}{d\theta^2}\right|_{\theta=\pi} = -ml^2\Omega^2 - mgl < 0 \text{ for all } \Omega.$$

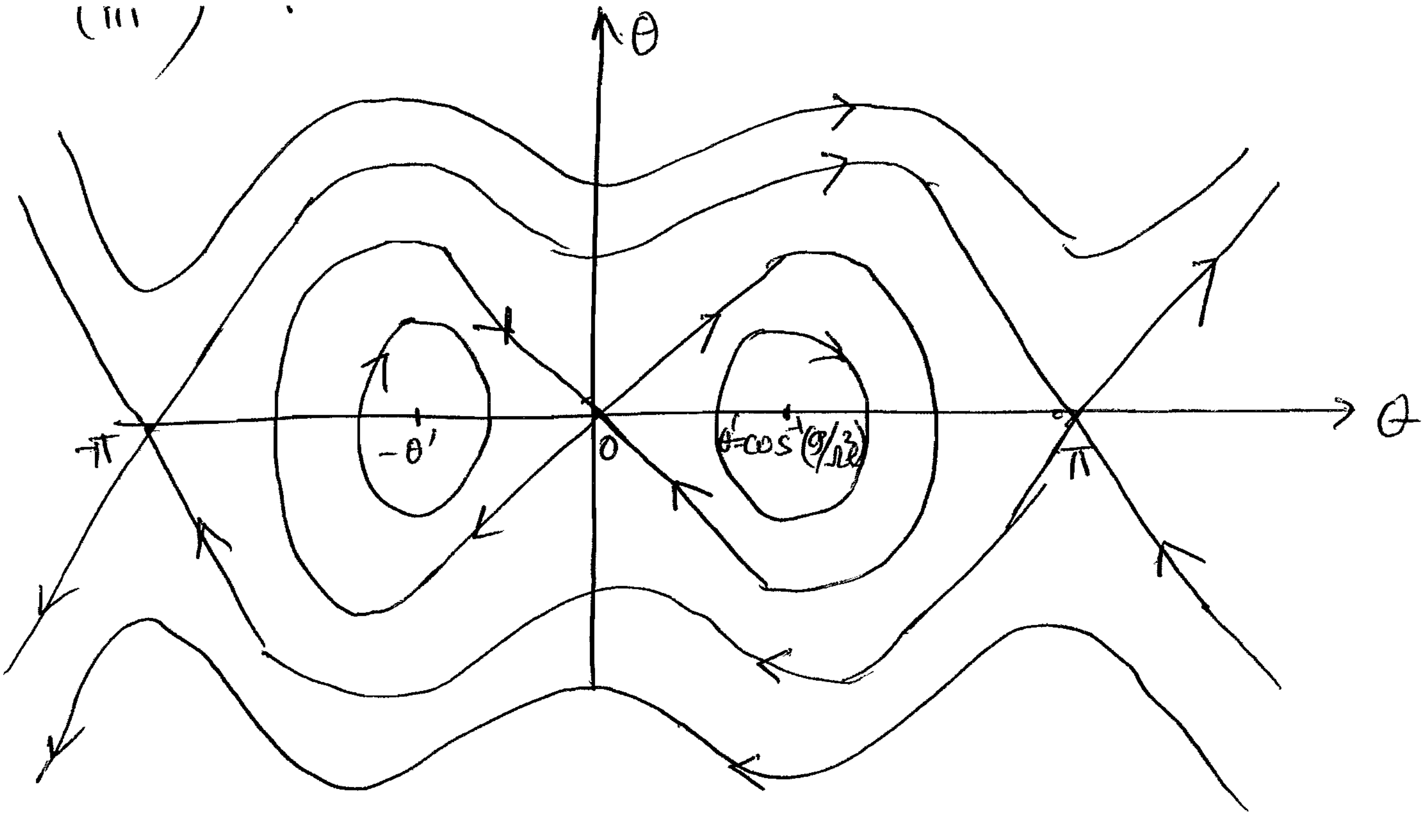
always unstable.

for $\Omega^2 > g/l$, $\left.\frac{d^2V_{eff}}{d\theta^2}\right|_{\theta=\cos^{-1}(g/\Omega^2 l)} = -\frac{mg^2}{\Omega^2} + m\Omega^2 l^2 > 0$ for $\Omega^4 > g^2/l^2$

$\therefore$ This point is stable "center"

[40%]

(iii)



This is the phase plane for

$\Omega^2 > g/l$; $-\pi \le \theta \le \pi$ [50%]