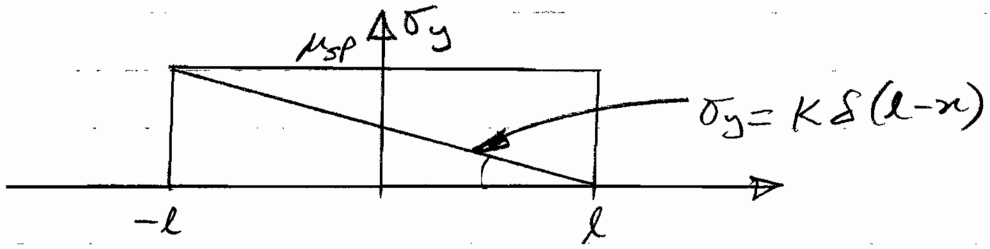
4C8 Dr D Cebon dc@eng.cam.ac.uk

4C8 SOLUTIONS 2006

1.



(a) Microslip first occurs at the rear of the contact area when

$$\sigma_y(x=-l) = \mu_s p$$

$$\Rightarrow \quad 2Kl\,\delta_{crit} = \mu_s p \Rightarrow \delta_{crit} = \frac{\mu_s p}{2Kl} \quad (1)$$

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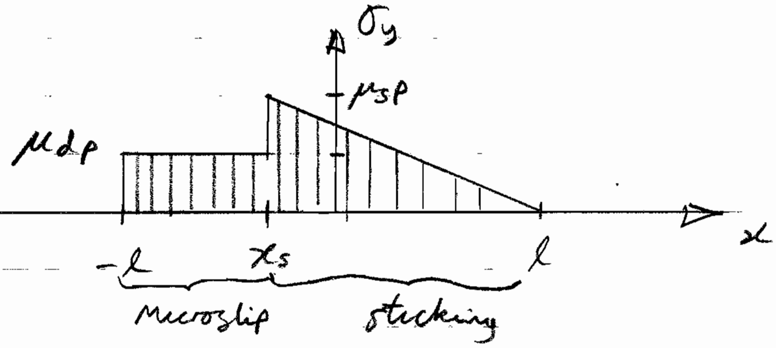
(b)



When $\delta > \delta_{crit}$, microslip first starts at x_s, where:

$$K\delta(l-x_s) = \mu_s p \Rightarrow x_s = l - \frac{\mu_s p}{K\delta} \quad (2)$$

The lateral force is given by area under $\sigma_y - x$

$$Y = \underbrace{2h\int_{-l}^{x_s} \mu_d p \, dx}_{\text{microslip}} + \underbrace{2h\int_{x_s}^{l} K\delta(l-x)\, dx}_{\text{no slip — linear region}}$$

$$= 2h\mu_d p(x_s+l) + 2hK\delta\left[-\tfrac{1}{2}(l-x)^2\right]_{x_s}^{l}$$

$$= 2h\mu_d p(x_s+l) + hK\delta(l-x_s)^2 \quad (3)$$

Substituting x_s from ② into ③ gives

$$Y = 2h\mu_d P\left(2l - \frac{\mu_s P}{K\delta}\right) + hK\delta\left(\frac{\mu_s P}{K\delta}\right)^2$$

$$= 4h\mu_d P l - \frac{2h\mu_d\mu_s P^2}{K\delta} + \frac{h\mu_s^2 P^2}{K\delta}$$

$$= 4h\mu_d P l + \frac{P^2 h}{K\delta}\mu_s(\mu_s - 2\mu_d) \quad \text{④}$$

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(check if $\mu_s = \mu_d = \mu$ $Y = 4h\mu P l - \frac{P^2 h}{K\delta}\mu^2$ ✓ as per lecture notes)

(c) For $\delta > \delta_{crit}$, Y is given by eq. 4.

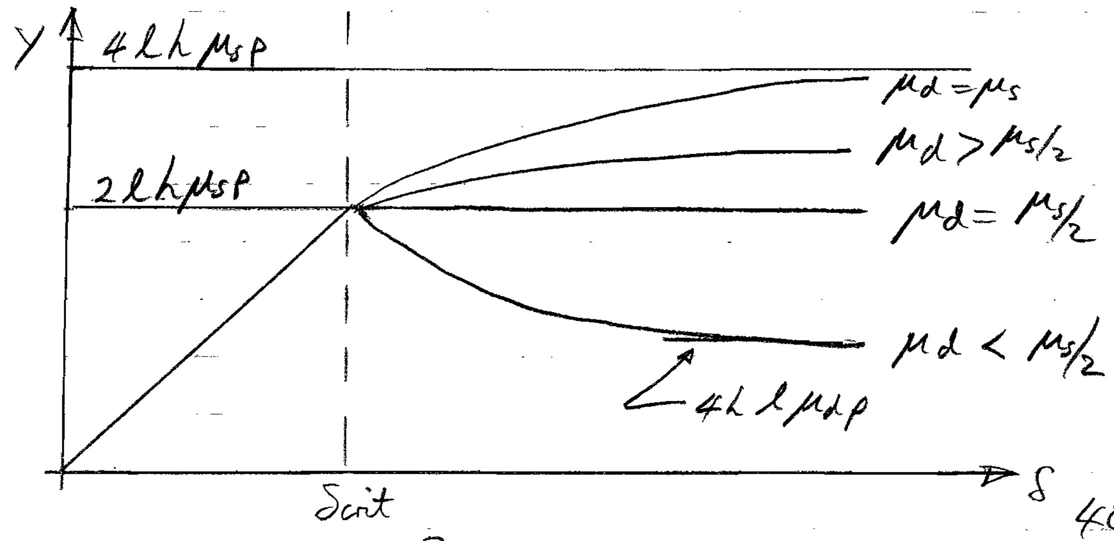
$$\Rightarrow \frac{dY}{d\delta} = -\frac{P^2 h}{K\delta^2}\mu_s(\mu_s - 2\mu_d)$$

$$\frac{dY}{d\delta} < 0 \quad \text{if} \quad \mu_s - 2\mu_d > 0$$

ie if $\mu_s > 2\mu_d \rightarrow \mu_d < \frac{\mu_s}{2}$

At $\delta = \delta_{crit}$, $x_s = -l$ & ③ $\Rightarrow Y = hK4l^2\delta_{crit}$
$= 2hl\mu_s P$

& at $\delta \to \infty$, ④ $\Rightarrow Y = 4hl\mu_d P$



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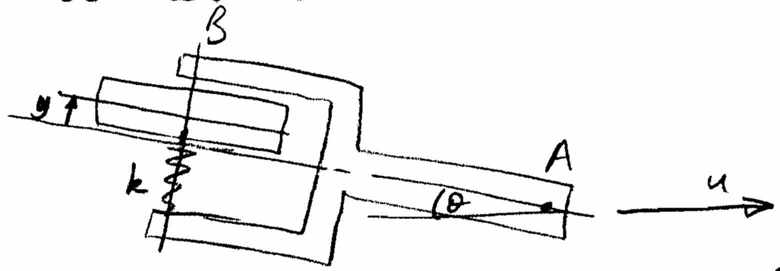
2a – See lecture notes

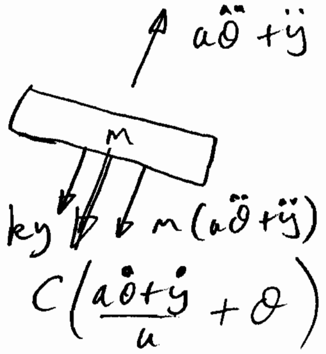
2b



Lateral velocity of wheel is: $a\dot{\theta} + \dot{y}$

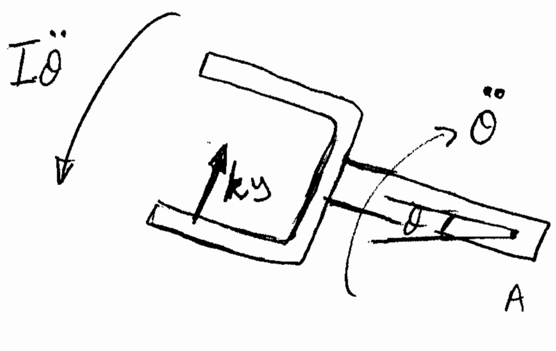
Lateral creep of wheel is $(a\dot{\theta} + \dot{y})/u + \theta$

Lateral tyre force is $C[(a\dot{\theta} + \dot{y})/u + \theta]$



$\underset{\sim}{\Sigma F}$

$$C\left(\frac{a\dot{\theta}+\dot{y}}{u} + \theta\right) + ky + m(a\ddot{\theta} + \ddot{y}) = 0 \qquad (1)$$



$\underset{\sim}{\Sigma M}$

$$I\ddot{\theta} - aky = 0 \qquad (2)$$

In matrix form:

$$\begin{bmatrix} M & ma \\ 0 & I \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} C/u & Ca/u \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & C \\ -ak & 0 \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = 0$$

characteristic equation:

$$\begin{vmatrix} Ms^2 + Cs/u + k & mas^2 + \frac{Cas}{u} + C \\ -ak & Is^2 \end{vmatrix} = 0$$

$$\left(ms^2 + \frac{Cs}{u} + k\right)(Is^2) - (-ak)\left(mas^2 + \frac{Cas}{u} + C\right) = 0$$

2 Cont

$$\underbrace{s^4(mI)}_{a_4} + \underbrace{s^3\left(\frac{cI}{u}\right)}_{a_3} + \underbrace{s^2\left(kI + ma^2k\right)}_{a_2} + \underbrace{s\left(\frac{ca^2k}{u}\right)}_{a_1} + \underbrace{cak}_{a_0} = 0$$

Routh Hurwitz: Stable if (I) All a's > 0

(II) $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2 a_0$

(I) is trivial $\rightarrow$ $u > 0$

(II) : $\frac{\cancel{c}a^2k}{\cancel{u}} \cdot \left(kI + ma^2k\right) \cdot \frac{\cancel{c}\cancel{I}}{\cancel{u}} > \left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)^2 m\cancel{I} + \left(\frac{\cancel{c}I}{\cancel{u}}\right)^{\cancel{2}} cak$

$\Rightarrow a^2k^2I + \cancel{a^4mk^2} > \cancel{a^4mk^2} + ICa\cancel{k}$

$\Rightarrow \underline{\underline{k > \frac{c}{a}}}$ is stability condition

3. (a) A two-axled vehicle with rigid suspension and tyres travelling along a sinusoidal profile of wavelength λ will experience pure bounce motion if a whole number of wavelengths $n\lambda$ fit within the wheelbase L (distance between front and rear axles). If there is a whole number plus a half wavelengths $(n+0.5)\lambda$ in the wheelbase L then the vehicle exhibits pure pitch motion and no bounce. The corresponding excitation frequencies f depend on vehicle speed V according to $V = f\lambda$. Thus the frequencies at which there is strong bounce excitation can be calculated using $f_{bounce} = nV/L$, and the frequencies at which there is strong pitch excitation are given by $f_{pitch} = (n+0.5)V/L$. For a vehicle with suspension, the wheelbase filtering leads to the vibration response of the sprung mass being strongly dependent on the speed of the vehicle.

(b) Simplify the model by eliminating the unsprung masses (they contribute little to the sprung mass modes of vibration), and add the tyre and suspension stiffness in series:

At the front the series stiffness is $k_{s1} = \dfrac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_{t1}}} = \dfrac{1}{\dfrac{1}{23}+\dfrac{1}{200}} = 20.6kN/m$

At the rear the series stiffness is $k_{s2} = \dfrac{1}{\dfrac{1}{k_2}+\dfrac{1}{k_{t2}}} = \dfrac{1}{\dfrac{1}{27}+\dfrac{1}{200}} = 23.8kN/m$

The sprung mass bounce natural frequency is then

$$f_{n\_bounce} = \frac{1}{2\pi}\sqrt{\frac{k_{s1}+k_{s2}}{m}} = \frac{1}{2\pi}\sqrt{\frac{20.6e3+23.8e3}{800}} = 1.19Hz$$

The pitch stiffness about the centre of mass is given by

$$\tau = k_{s1}a^2 + k_{s2}b^2 = 20.6e3(1.35)^2 + 23.8e3(1.55)^2 = 94.8kNm/rad$$

so the sprung mass pitch natural frequency is

$$f_{n\_pitch} = \frac{1}{2\pi}\sqrt{\frac{\tau}{J}} = \frac{1}{2\pi}\sqrt{\frac{94.8e3}{1350}} = 1.33Hz$$

(c) The transfer functions in Fig. 2 are for road velocity input and sprung mass acceleration output, so compared to the Mechanics Data Book case (c), the magnitudes of the transfer function will be scaled by $j\omega$. Thus we expect zero response at zero frequency and a flat response at frequencies above the resonance frequency. This can be seen as the constant height of peaks at frequencies above the sprung mass resonant frequencies (1-2Hz) and before the unsprung mass modes come into play at about 10Hz.

Next, consider the effect of wheelbase filtering using the information given in the answer to part (a). Calculate the frequencies at which there is strong bounce excitation and weak pitch excitation (whole number of wavelengths in the wheelbase) $f_{bounce} = nV/L$:

Table 1

n	0	1	2	3
V=10m/s	0 Hz	3.5 Hz	6.9 Hz	10.4 Hz
V=30m/s	0 Hz	10.4 Hz	22.4 Hz	

These frequencies clearly match the 'troughs' observed in the pitch transfer functions.

Calculate the frequencies at which there is strong pitch excitation and weak bounce excitation (whole number plus a half wavelengths in the wheelbase) $f_{pitch} = (n + 0.5)V/L$:

Table 2

n	0.5	1.5	2.5	3.5
V=10m/s	1.7 Hz	5.2 Hz	8.6 Hz	12.1 Hz
V=30m/s	5.2 Hz	15.5 Hz	25.9 Hz	

These frequencies clearly match the 'troughs' observed in the bounce transfer functions.

In the region of the sprung mass natural frequencies at about 1Hz to 1.5Hz, the strong pitch response (and small bounce response) at 10m/s is explained by the proximity of the sprung mass natural frequencies to the 1.7Hz frequency in table 2. The strong bounce response (and small pitch response) at 30m/s is explained by the proximity of the sprung mass natural frequencies to 0Hz in table 1.

The unsprung mass modes of vibration are at approximately

$$\frac{1}{2\pi}\sqrt{\frac{k_t}{m_u}} = \frac{1}{2\pi}\sqrt{\frac{200e3}{45}} = 10.6Hz$$

The road to sprung mass transfer functions generally reduce in magnitude above this frequency because the unsprung masses act to filter the excitation from the road.

(d)

- Optimum values of stiffness and damping are likely to depend on speed, due to the wheelbase filtering mechanism. The engineer will need to consider performance at a range of speeds and decide a suitable compromise. Alternatively a suspension that can adapt its damping and/or stiffness with vehicle speed might be considered.
- The engineer will need to decide how to weight the pitch and bounce acceleration responses. It is common practice to minimise pitch at the expense of bounce, but there is still much to understand in this area. Most engineers rely heavily on subjective testing of the vehicle.
- The pitch plane model neglects lateral and roll degrees of freedom. Generally accelerations in these directions are small. Low vehicle speeds (when lateral tyre stiffness is significant) and/or discrete inputs from the road (which may have significant roll component) might require a lateral-roll plane model.

4 (a) See lecture notes. Considering steady-state lateral behaviour of the tyre, slip angle is α, so sideslip velocity is αV. Side force F is slip angle α times cornering stiffness C. So equivalent lateral damping c_{lat} is that which gives force $F = \alpha C$ at sideslip velocity αV, thus $c_{lat} = \alpha C / \alpha V = C/V$.

(b) At very low speed, c_{lat} becomes very large, so assume it becomes rigid. Newton's 2nd law on the inertia:

$$I\ddot{\theta} + b\dot{\theta} + \tau\theta - mh\ddot{y} = \tau\phi + b\dot{\phi}$$

Newton's 2nd law on the mass

$$m\ddot{y} + k_{lat}y = -k_{lat}h\theta$$

In matrix form:

$$\begin{bmatrix} I & -mh \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} \tau & 0 \\ k_{lat}h & k_{lat} \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}\phi + \begin{bmatrix} b \\ 0 \end{bmatrix}\dot{\phi}$$

For zero damping and free response:

$$\begin{bmatrix} I & -mh \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} \tau & 0 \\ k_{lat}h & k_{lat} \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Laplace transform, zero initial conditions, and replacing s with $j\omega$ gives

$$-\omega^2\begin{bmatrix} I & -mh \\ 0 & m \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} + \begin{bmatrix} \tau & 0 \\ k_{lat}h & k_{lat} \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Non-trivial solutions given when:

$$\begin{vmatrix} \tau - I\omega^2 & mh\omega^2 \\ k_{lat}h & k_{lat} - m\omega^2 \end{vmatrix} = 0$$

or

$$\omega^4 mI - \omega^2\left(m\tau + k_{lat}I + mh^2k_{lat}\right) + k_{lat}\tau = 0 \qquad (1)$$

(c) From Fig. 2b, suitable values for the simple model are:

$m = 800$ kg
$I = 460$ kg m^2
$h = 0.5$m

$$\tau = \frac{2T^2}{\frac{1}{k_s} + \frac{1}{k_t}} = \frac{2 \times 0.75^2}{\frac{1}{200e3} + \frac{1}{30e3}} = 29.35 kNm/rad$$

$k_{lat} = 400$ kN/m

Putting these values into (1) and solving for ω^2 gives natural frequencies of $\omega_1 = 1.05$ Hz and $\omega_2 = 4.32$ Hz.

(d) At very high speed the lateral tyre damping tends to zero. Equation (1) is applicable if the lateral stiffness term k_{lat} is set to zero. The equation then becomes

$$\omega^4 mI - \omega^2 m\tau = 0$$

Solutions are $\omega_1 = 0$ and $\omega_2 = \sqrt{\frac{\tau}{I}}$. The zero frequency mode is a 'rigid body' mode, and indicates that the vehicle will readily adopt a lateral displacement after the application of an external lateral impulse, such as sidewind gust.

The roll mode natural frequency at ω_2 (=1.27 Hz) may not be observed in measured responses of the sprung mass roll motion, because at high speed the roll input from a randomly rough road surface at this frequency is very small compared to the vertical input.