Part IIB 2006 Module 4C9 Solutions.

1 (a) (i) to find $\underline{a} \cdot (\underline{b} \times \underline{c})$ in subscript notation

$$\underline{b} \times \underline{c} = e_{ijk} b_j c_k$$

$$\underline{u} \cdot \underline{v} = u_i v_i$$

$$\therefore \quad \underline{a} \cdot (\underline{b} \times \underline{c}) = a_i e_{ijk} b_j c_k$$

$$= e_{ijk} a_i b_j c_k$$

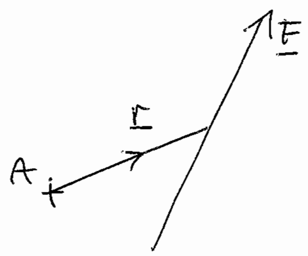
$$e_{ijk} a_i b_j c_k$$

$$= e_{123} a_1 b_2 c_3 + e_{231} a_2 b_3 c_1 + e_{312} a_3 b_1 c_2$$

$$+ e_{132} a_1 b_3 c_2 + e_{213} a_2 b_1 c_3 + e_{321} a_3 b_2 c_1$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

$$- a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$



So if

$$\underline{c} = \underline{F} = (3, 2, 4)$$

$$\underline{b} = \underline{r} = (2, 1, 3)$$

$$\underline{a} = \underline{e} = \frac{1}{\sqrt{53}}(1, 4, 6)$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{e} \cdot (\underline{r} \times \underline{F}) = \Big(1.1.4 + 4.3.3 + 6.2.2$$

$$- 1.3.2 - 4.2.4 - 6.1.3\Big)\frac{1}{\sqrt{53}}$$

$$= \frac{1}{\sqrt{53}}(4 + 36 + 24 - 6 - 32 - 18)$$

$$= \frac{8}{\sqrt{53}} \text{ Nm}$$

(ii) $$v_k = e_{kji} u_{ij}$$

$$e_{k\ell m} v_k = e_{k\ell m} e_{kji} u_{ij}$$
$$= (\delta_{\ell j}\delta_{mi} - \delta_{\ell i}\delta_{mj}) u_{ij}$$
$$= \delta_{\ell j}\delta_{mi} u_{ij} - \delta_{\ell i}\delta_{mj} u_{ij}$$

i.e. $$e_{k\ell m} v_k = u_{m\ell} - u_{\ell m}$$

But if u_{ij} is skew-symmetric

$$u_{ij} = -u_{ji}$$

$$\therefore\ e_{k\ell m} v_k = 2u_{m\ell}$$

i.e. $$u_{m\ell} = \tfrac{1}{2} e_{k\ell m} v_k$$

or $$u_{ij} = \tfrac{1}{2} e_{kji} v_k$$

1. (b)

(i)
$$\varepsilon_y = \varepsilon_z = -\mu\,\varepsilon_x \qquad \text{(total strains)}$$
$$\varepsilon_y^e = \varepsilon_z^e = -\nu\,\varepsilon_x^e \qquad \text{(elastic strains)}$$
$$\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p = 0 \qquad \text{(plastic incompressibility)}$$

$$\Rightarrow \quad \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_x - 2\mu\varepsilon_x = (1-2\mu)\varepsilon_x$$

But $\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_x^e + \varepsilon_y^e + \varepsilon_z^e = (1-2\nu)\,\varepsilon_x^e$

$$\Rightarrow \quad \varepsilon_x = \frac{1-2\nu}{1-2\mu}\,\varepsilon_x^e$$

Now, $\sigma_x = E_{sec}\,\varepsilon_x = E\,\varepsilon_x^e \quad \Rightarrow \quad \varepsilon_x = \frac{E}{E_{sec}}\,\varepsilon_x^e$

$$\Rightarrow \quad \frac{1-2\nu}{1-2\mu} = \frac{E}{E_{sec}}$$

Re-arranging $\Rightarrow \quad \mu = \frac{1}{2} - \frac{E_{sec}}{2E}(1-2\nu)$ #

(ii) $\varepsilon_{ij}^p = \frac{3}{2}\left(\frac{1}{E_{sec}} - \frac{1}{E}\right) s_{ij} \qquad$ (J_2 deformation theory)

$$\Rightarrow \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] + \frac{3}{2}\left(\frac{1}{E_{sec}} - \frac{1}{E}\right)\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

$$= \frac{1}{E}\left(\frac{2\nu-1}{2}\sigma_{ij} - \frac{2\nu-1}{2}\sigma_{kk}\delta_{ij}\right) + \frac{3}{2E_{sec}}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

$$\xrightarrow{\text{(i) result}} = \frac{2\mu-1}{2E_{sec}}\left(\sigma_{ij} - \sigma_{kk}\delta_{ij}\right) + \frac{3}{2E_{sec}}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

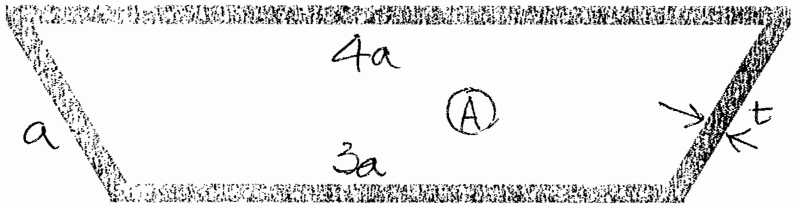
$$= \frac{1}{E_{sec}(\sigma_e)}\left[(1+\mu)\sigma_{ij} - \mu\sigma_{kk}\delta_{ij}\right]$$

#

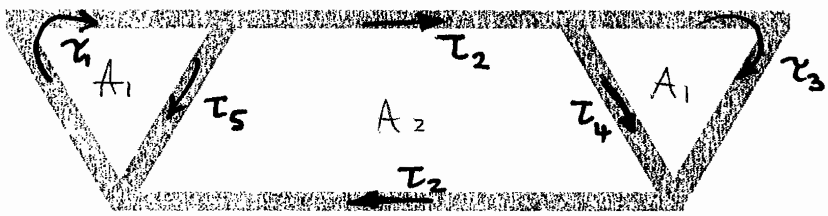
For Box section enclosed area $A = \frac{7}{2}a \cdot \frac{\sqrt{3}}{2}a = \frac{7\sqrt{3}a^2}{4}$

$$\oint \frac{ds}{dt} = \frac{9a}{t} \qquad \therefore \frac{T}{G\alpha} = \frac{4A^2}{\oint ds/t} = \frac{4 \times 49.3a^4 t}{16 \times 9a}$$

$$\therefore \; T/G\alpha = 49/12\, a^3 t = 4.08\, a^3 t$$



(a)



(b)

$$\begin{cases} 2GA_1\alpha = \tau_1 2a + \tau_5 a \\ 2GA_2\alpha = \tau_2 . 2a + \tau_4 a + \tau_2 . 3a - \tau_5 a \\ 2GA_1\alpha = \tau_3 2a - \tau_4 . a \end{cases}$$

assume $d_1 > d_2 > d_3$ effective membrane displacement

$$\begin{cases} \tau_1 = \frac{d_1}{t} \quad \tau_2 = \frac{d_2}{t} \quad \tau_5 = \frac{d_1 - d_2}{t} \quad \tau_4 = \frac{d_2 - d_3}{t} \\ \tau_3 = \frac{d_3}{t} \end{cases}$$

shear flows

$$\frac{2GA\alpha}{a} = 2\tau_1 + \tau_5 = 2\tau_3 - \tau_4$$

But $\tau_1 - \tau_2 = \tau_5 \qquad \tau_2 - \tau_3 = \tau_4$ all/t.

$$\therefore \; 2\tau_1 + \tau_1 - \tau_2 = 2\tau_3 - \tau_2 + \tau_3$$

$$\therefore \; \tau_1 = \tau_3 \quad \text{symmetry}$$

$A_1 = \frac{\sqrt{3}a^2}{4}$; $A_2 = \frac{5\sqrt{3}a^2}{4}$

$$\therefore \begin{cases} 2G\frac{\sqrt{3}}{4}a\cancel{t}\alpha = 2\tau_1 + \tau_1 - \tau_2 \\ 2G5\frac{\sqrt{3}}{4}a\alpha = 2\tau_2 + \tau_2 - \tau_1 + 3\tau_2 - \tau_1 + \tau_2 \end{cases}$$

$$\begin{cases} 3\tau_1 - \tau_2 = \frac{\sqrt{3}}{2}Ga\alpha \\ -2\tau_1 + 7\tau_2 = \frac{5\sqrt{3}}{2}Ga\alpha \end{cases}$$

$$\begin{cases} 6\tau_1 - 2\tau_2 = \sqrt{3}Ga\alpha \\ -6\tau_1 + 21\tau_2 = \frac{15}{2}\sqrt{3}Ga\alpha \end{cases}$$

$\therefore \tau_2 = \frac{17\sqrt{3}}{19.2}Ga\alpha$

and

$\tau_1 = \frac{12\sqrt{3}}{19.2}Ga\alpha$

But $T = 2A_1d_1 + 2A_2d_2 + 2A_1d_3$

$$= 2A_1\tau_1 t + 2A_2\tau_2 t + 2A_1\tau_1 t \qquad (\tau_3 \equiv \tau_1)$$

$$= 4 \cdot \frac{\sqrt{3}}{4}a^2t\frac{12\sqrt{3}}{19.2}Ga\alpha + 2\frac{5\sqrt{3}}{4}a^2t\frac{17\sqrt{3}}{19.2}Ga\alpha$$

$$= \left(\frac{12.3}{19.2} + \frac{\cancel{10}^{5}.17.3}{2\cancel{4}.19.2}\right)Ga^3t\alpha$$

$$\therefore \frac{T}{G\alpha} = \frac{72 + 15.17}{19.4}a^3t = \frac{327}{76}a^3t \Rightarrow 4.303a^3t$$

So increase in stiffness $\frac{4.303}{4.08} = 5.5\%$

(b) Fully plastic torque can be evaluated by sand-hill analogy.

Consider shaft with section ABCD in which the inscribed circle just touches all sides, as in (i) & (ii)

Then pyramid formed by "sand" has apex of height h where $h = k \times R$

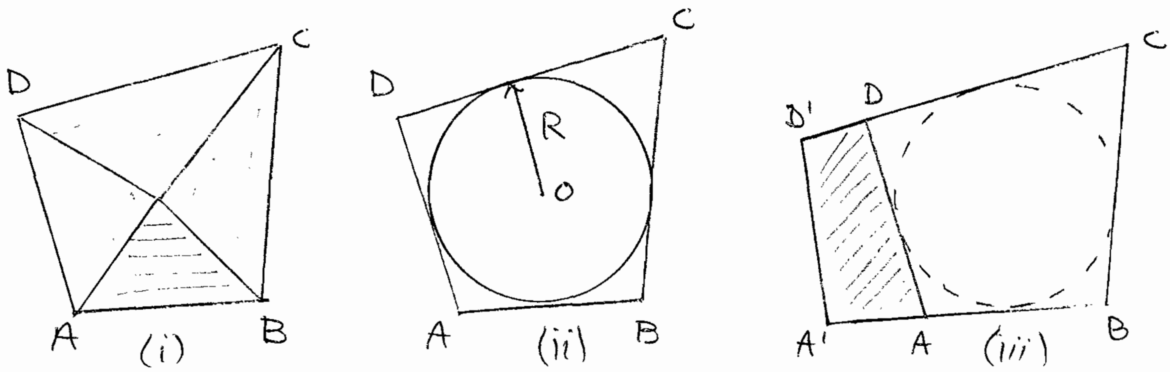
Torque $\Rightarrow$ 2 volume of pyramid

But since volume of pyramid is $\frac{1}{3} \times \text{base} \times \text{height}$

$$\text{Torque } T_p = 2 \times \tfrac{1}{3} A_o \times k \times R$$

$$T_p = \tfrac{2}{3} A_o R k$$

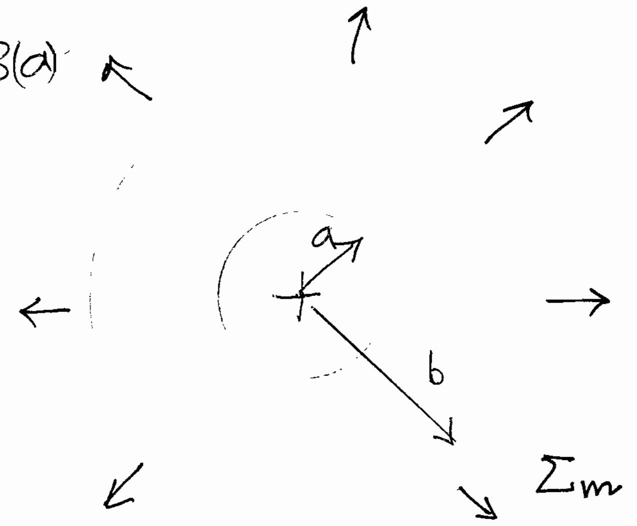
So expression is exact.



But adding material cannot reduce plastic resistance so that fully plastic torque T_p' for A'ABCDD' must be greater than that for ABCD

ie. $T_p = \frac{2}{3} A_o R k$ is a lower bound on T_p'

3(a)



$$f_v = \frac{\pi a^2}{\pi b^2}$$

So $b = \dfrac{a}{\sqrt{f_v}}$

$$u^* = Ar + \frac{B}{r}$$

$u^* \Rightarrow 0$ at $r = a$ $\therefore$ $0 = Aa + \dfrac{B}{a}$

$$B = -Aa^2$$

$$\therefore \quad u^* = Ar\left(1 - \frac{a^2}{r^2}\right) = Ar - \frac{Aa^2}{r}$$

Strain rates $\quad \dot{\varepsilon}_r^* = \dfrac{du^*}{dr} = A + \dfrac{Aa^2}{r^2} = A\left(1 + \dfrac{a^2}{r^2}\right)$

$v^* \Rightarrow 0$ $\quad \dot{\varepsilon}_\theta^* = \dfrac{u^*}{r} = A\left(1 - \dfrac{a^2}{r^2}\right)$

$$\dot{\varepsilon}_e^* = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^*\dot{\varepsilon}_{ij}^*} = \sqrt{\frac{2}{3}\left(\dot{\varepsilon}_r^{*2} + \dot{\varepsilon}_\theta^{*2}\right)}$$

$$= A\sqrt{\frac{2}{3}}\left\{1 + \frac{2a^2}{r^2} + \frac{a^4}{r^4} + 1 - \frac{2a^2}{r^2} + \frac{a^4}{r^4}\right\}^{1/2}$$

$$\dot{\varepsilon}_e^* = \frac{2A}{\sqrt{3}}\left\{1 + \frac{a^4}{r^4}\right\}^{1/2}$$

Now UB theorem $\quad \displaystyle\int_S T_i^L u_i^* \, dS \leq \int_V \sigma_y^M \dot{\varepsilon}_e^* \, dV$

So that

$$\Sigma_M^L \, u_b^* \, 2\pi b \leq \int_a^b \sigma_y^M \cdot \frac{2A}{\sqrt{3}} \left\{1+\frac{a^4}{r^4}\right\}^{1/2} . 2\pi r \, dr$$

$$u_b^* = Ab(1-a^2/b^2)$$

$$\therefore \Sigma_M^L \cdot \cancel{A}b(1-a^2/b^2)\cancel{2\pi}b$$

$$\leq \frac{2\cancel{A}}{\sqrt{3}} \cdot \cancel{2\pi} \, \sigma_y^M \int_a^b r\left\{1+\frac{a^4}{r^4}\right\}^{1/2} dr$$

$$\frac{\Sigma_M^L}{\sigma_y^M} \cdot \frac{\sqrt{3}}{2} b^2(1-a^2/b^2) \leq \underbrace{\int_a^b r\left\{1+\frac{a^4}{r^4}\right\}^{1/2} dr}_{I}$$

$$I = \frac{a^2}{2}\int_{\theta_1}^{\theta} \sec^2\theta \csc\theta \, d\theta \qquad \begin{cases} \tan\theta_1 = 1 \; ; \; \theta_1 = \pi/4 \\ \tan\theta_2 = b^2/a^2 = 1/f_v \end{cases}$$

$$\therefore \frac{2I}{a^2} = \Big[\csc\theta \tan\theta\Big]_{\theta_1}^{\theta_2} + \int_{\theta_1}^{\theta_2} \tan\theta \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

$$\text{i.e. } \frac{2I}{a^2} - \Big[\sec\theta\Big]_{\theta_1}^{\theta_2} = \int \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin^2\theta} \, d\theta = \int_{\theta_1}^{\theta_2} \csc\theta \, d\theta$$

i.e.

$$\frac{2I}{a^2} - \sec\theta_2 + \sec\theta_1 = \Big[\ln\left(\tan\theta/2\right)\Big]_{\theta_1}^{\theta_2}$$

$$\therefore \frac{\Sigma_M^L}{\sigma_y^M} \frac{\sqrt{3}}{\cancel{2}} b^2(1-a^2/b^2) \cdot \frac{\cancel{2}}{a^2} = \sec\theta_2 - \sec\theta_1 + \ln\left[\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right]$$

But $\frac{a^2}{b^2} = f_v$

$$\therefore \quad \frac{\sqrt{3}\,\Sigma_M^L}{\sigma_y^M} \cdot \frac{1-f_v}{f_v} = \sqrt{1 + (1/f_v)^2} - \sqrt{2} + \ln\left[\frac{\tan(\theta_2/2)}{\tan(\pi/8)}\right]$$

If $f_v \Rightarrow 0$ then

$$\frac{\sqrt{3}\,\Sigma_M^L}{\sigma_y^M} \cdot \frac{1}{f_v} \simeq \frac{1}{f_v}$$

i.e. $\Sigma_M^L = \frac{\sigma_y^M}{\sqrt{3}}$

If $f_v \Rightarrow 1$

So $\Sigma_M^L \Rightarrow \frac{f_v}{1-f_v}(\;) \Rightarrow \infty$ no yield ✓

(b) $u^* = \left(Cr + \frac{D}{r}\right)\sin 2\theta; \quad v^* = \left(Cr + \frac{D}{r}\right)\cos 2\theta$

again at $r = a \quad u^* = v^* \Rightarrow 0$

$\therefore \quad D = -Ca^2$

So $u^* = Cr\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta; \quad v^* = Cr\left(1 - \frac{a^2}{r^2}\right)\cos 2\theta$

$$\varepsilon_r^* = \frac{\partial u^*}{\partial r} = C\left(1 + \frac{a^2}{r^2}\right)\sin 2\theta$$

$$\varepsilon_\theta^* = \frac{u^*}{r} + \frac{1}{r}\frac{\partial v^*}{\partial \theta} = C\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta - 2C\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta$$

$$= -C\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta$$

$$\gamma_{r\theta}^* = \frac{\partial v^*}{\partial r} + \frac{1}{r}\frac{\partial u^*}{\partial \theta} - \frac{v^*}{r}$$

$$\Rightarrow \quad C\left(1 + \frac{a^2}{r^2}\right)\cos 2\theta + 2C\left(1 - \frac{a^2}{r^2}\right)\cos 2\theta - C\left(1 - \frac{a^2}{r^2}\right)\cos 2\theta$$

$$= C\left[1 + \cancel{\frac{a^2}{r^2}} + 2 - \cancel{2\frac{a^2}{r^2}} - 1 + \cancel{\frac{a^2}{r^2}}\right]\cos 2\theta$$

$$\gamma_{r\theta}^* = 2C\cos 2\theta$$

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