Q1

In the absence of significant adhesion the pressure distribution between an elastic sphere and plane can be described by the semi-elliptical Hertz equation

$$p(r) = p_0\left\{1-(r/a)^2\right\}^{1/2} \quad \text{where } p_0 = \frac{3P_0}{2\pi a^2}$$

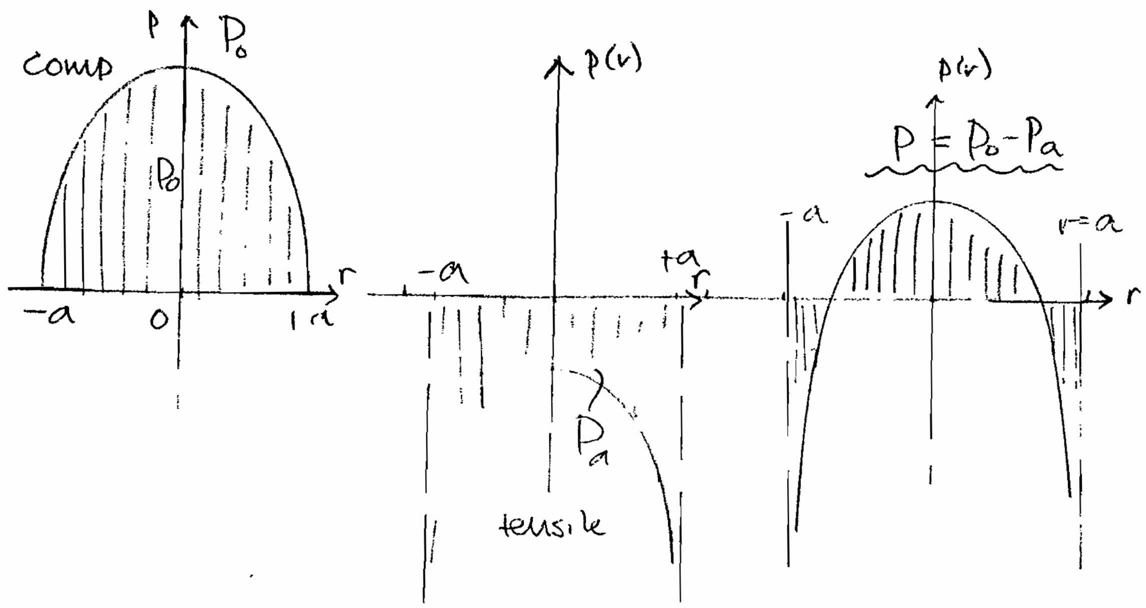
a is the radius of contact patch, and P_0 the total applied compressive load.

In the presence of significant adhesion driven by surface energy the 2 solids are effectively pulled together by some additional surface traction. This must be such as to give every point some distance from the interface the same normal displacement. Thus this additional effect must just be the inverse of the pressure under a rigid flat punch.

Because the material is linearly elastic the two distributions can simply be added.

i.e. $$p(r) = \underbrace{\frac{3P_0}{2\pi a^2}\left(1-(r/a)^2\right)^{1/2}}_{\text{Hertz comp.}} - \underbrace{\frac{P_a}{2\pi a^2}\left(1-(r/a)^2\right)^{-1/2}}_{\text{adhesive tensile.}}$$

When $r=a$ $p(r) \Rightarrow \infty$ tension



The balance between these two systems and hence the relation between load and contact area can be found either by a direct energy argument or by the method suggested.

If $K_I = \frac{P_a}{2a\sqrt{\pi a}}$ ~~another~~ and in limit $\frac{K_I^2}{2E^*} = w$

then $P_a = 2a\sqrt{\pi a}\, K_I = 2a\sqrt{\pi a}\sqrt{2E^* w}$

But $P = P_0 - P_a$

and $P_0 = \frac{4E^* a^3}{3R}$ for Hertz

$\therefore \quad P = \frac{4E^* a^3}{3R} - 2a\sqrt{2E^* w \pi a}$

or $\quad P = \frac{4E^* a^3}{3R} - \sqrt{8\pi a^3 w E^*} \quad$ (JKR)

At pull off in a displacement controlled machine

$\frac{dP}{da} \Rightarrow 0 \quad$ ie. $\frac{4E^* a^2}{R} - \frac{3}{2} a^{1/2}\sqrt{8\pi w E^*} = 0$

ie. $\frac{16E^{*2}}{R^2} a^3 = \frac{9}{4} \cdot 8\pi w E^*$

ie. $a = \left(\frac{9\pi R^2 w}{8E^*}\right)^{1/3}$

So if $E^* = 37\,GPa, \quad w = 1.3\,Jm^{-2}$

and since $R^{-1} = 1/8 + 1/8$ ie. $R = .004\,m$

$a = \left[\frac{9 \times \pi \times .004^2 \times 1.3}{8 \times 37 \times 109}\right]^{1/3} = 12.5 \times 10^{-6}\,m$

$P = \frac{4 \times 37 \times 10^9 \times (12.5 \times 10^{-6})^3}{3 \times .004} - \sqrt{8 \times \pi \times (12.5 \times 10^{-6})^3 \times 1.3 \times 37 \times 10^9}$

$= \quad -24.5 \times 10^{-3}\,N \quad$ ie. 24.5 mN tensile

(a) $k = \dfrac{\pi^2 HWE}{4L}$; $b = \dfrac{m\omega}{Q}$

$k = 1.97 \times 10^5$ N/m ; $b = 1.516 \times 10^{-7}$ kg/s

$m = \rho HWL$

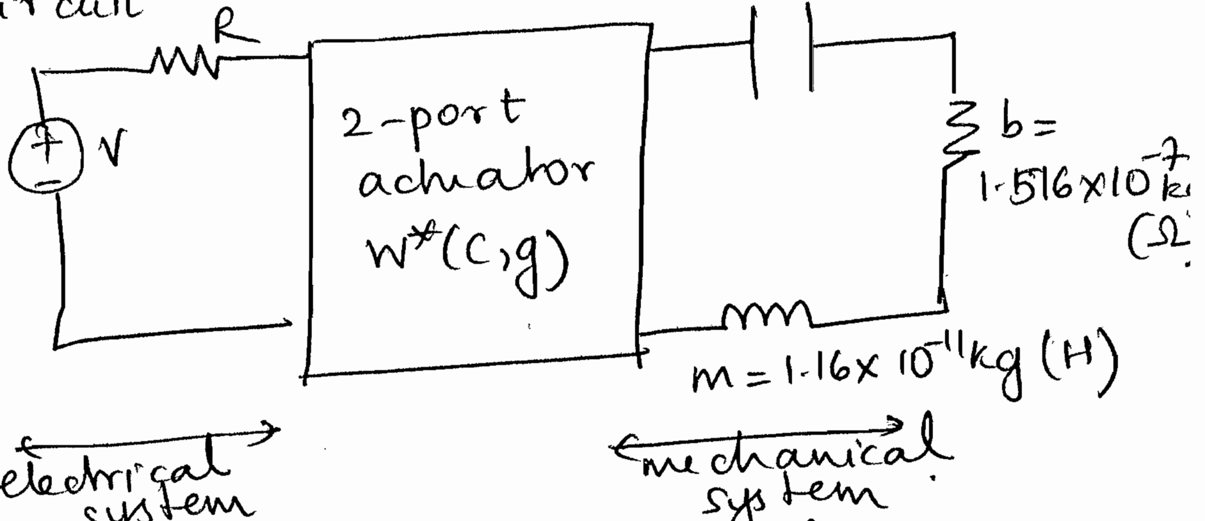
$m = 1.16 \times 10^{-11}$ kg

$\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1.97 \times 10^5}{1.16 \times 10^{-11}}}$

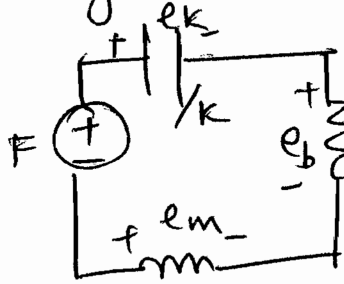
$\omega = 1.3 \times 10^8$ rad/sec | natural frequency.

$f = 20.7$ MHz

(b) A lumped equivalent electrical circuit is



mechanical circuit by KVL



$F - e_k - e_b - e_m = 0$

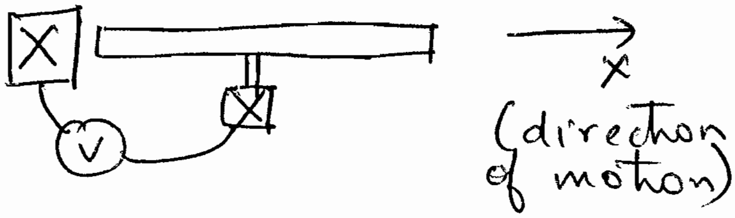
$F - kx - b\dot{x} - m\ddot{x} = 0$

$\dfrac{X(s)}{F(s)} = \dfrac{1}{k + bs + ms^2}$

where m, k and b are as calculated in (a)

(c) Stored energy in the capacitor

$W = \frac{1}{2} C V^2$



$$W = \frac{1}{2}\frac{V^2 \epsilon_0 HW}{g+x}$$

the electrostatic force is given by

$$F = -\frac{\partial W}{\partial x}$$

$$F = \frac{1}{2}\frac{V^2 \epsilon_0 HW}{(g+x)^2}$$

This is a parallel plate actuation mechanism with a critical point reached for a pull-in instability when the motion of the structure exceeds $1/3$rd of the gap.

(d) For a small signal AC excitation

$$F = \frac{\epsilon_0 V_{DC} V_{ac} (HW)}{g^2} \quad \text{for } x \ll g$$

$$F = 1.77 \text{ nN}$$

$$x = F/k \cdot Q$$

$$x = 89.7 \text{ pm} \ll 0.5 \mu m$$

as previously assumed.

3. Drive deflection x.

(a) Stored energy in capacitive actuator (W)

$$W = \frac{1}{2} C V^2 = \frac{1}{2} \frac{N \epsilon_0 H (L-X)}{g} V^2$$

$$F = -\frac{\partial W}{\partial x}$$

$$F = \frac{1}{2} N \epsilon_0 \frac{H}{g} V^2$$

(b) $k_x = k_y = \frac{5}{2} E H \left(\frac{W}{L}\right)^3 = 16 \text{ N/m}$

$$X_{DC=10V} = \frac{1}{2} (400) \times 8.85 \times 10^{-12} \times \left(\frac{5}{1}\right) \times \frac{100}{16}$$

$$= 55.3 \text{ nm}$$

(c) $x_{a=1g} = y_{a=1g} = \frac{m \times g}{k} = 1.78 \text{ nm}$

(d) Parallel plate electrodes: typically higher sensitivity per unit farad of sense capacitance. This is a nonlinear transduction mechanism and is limited by pull-in instability.

Comb drive electrodes: less sensitivity but a more linear response and the sideways instability voltag are typically much larger than the equivalent parallel-plate case.

(e) $\omega_{drive} = \sqrt{\frac{K}{m}} \Rightarrow 11.8\ kHz \approx 77500\ Hz$

$= \omega_{sense}$

Hence, we get expression for displacement along sense direction

$$y_{sens} = F_{coriolis} / k_y$$

$$= \frac{2 \times m \times \Omega_z \times \omega_x \times X}{k_y}$$

$$y_{sens} = 0.135\ nm$$

(f) Improved sensitivity can be achieved by matching the drive and sense frequencies. The amplitude of the response along the sense (y) direction is roughly amplified by the Q-factor. However the bandwidth is limited to the 3 dB bandwidth of the sense mode response.

4 (a) From structures databook

$$w_{max@tip} = \frac{FL^3}{3EI}$$

$$F = w_{max} \cdot 3E\left(\frac{WH^3}{12}\right) \cdot \frac{1}{L^3}$$

$$F = \frac{0.2 \times 5 \times 10^{-12} \times 160 \times 10^9}{4} \left(\frac{2}{500}\right)^3$$

$$F = 2.56\,nN$$

(b) Max. bending moment $= \frac{FL}{EI}$ (structu databoo

$$\sigma_{max} = \frac{H}{2} \cdot E \cdot \frac{FL}{EI} = \frac{6L}{H^2 W} \cdot F$$

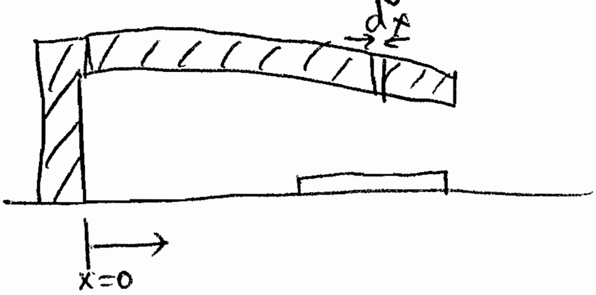
$$\sigma_{max} = \frac{6 \times 500}{5} \times \frac{2.56 \times 10^{-9}}{(2 \times 10^{-6})^2}$$

$$= 384\ kPa$$

$$\frac{\Delta R}{R} = \pi_l \sigma_{max} = -100 \times 10^{-11} \times 3.84 \times 1($$

$$\frac{\Delta R}{R} = 3.84 \times 10^{-4}$$ (fractional change in resistance)

(c) $C_{nominal} = \frac{\epsilon_0 A}{g}$ and ΔC for a chang in capacitance



$$= \frac{\epsilon_0 W dx}{g - w(x)} - \frac{\epsilon_0 W L_{ov}}{g}$$

$L_{ov} = 50\,\mu m$

$$\Delta C_{net} = \int_{450\mu m} \Delta C \, dx$$

Max tip deflection $\underset{(200nm)}{w} \ll g$ $(2\mu m)$

Hence

$$\Delta C_{net} \simeq \int_{450\mu m}^{500\mu m} \frac{\varepsilon_0 W}{g} \left(\frac{w(x)}{g} \right) dx$$

$w(x)$ is given by :—

$$\frac{d^2 w}{dx^2} = \frac{F}{EI}(L - x)$$ (Structures databook)

and $w(0) = 0$ and $\left.\frac{dw}{dx}\right|_{x=0} = 0$

$$\therefore \; w(x) = \frac{FL}{2EI} x^2 \left(1 - \frac{x}{3L} \right)$$

substituting into expression for ΔC_{net}

$$\Delta C_{net} = \frac{\varepsilon_0 W}{g^2} \left[\frac{FLx^3}{6EI} - \frac{Fx^4}{24EI} \right]_{450\mu m}^{500\mu m}$$

$$\frac{\Delta C_{net}}{C_{nom}} = \frac{1}{L_{ov}\, g} \cdot \frac{F}{6EI} \left[Lx^3 - \frac{x^4}{4} \right]_{450\mu m}^{500\mu m}$$

Fractional change in capacitance $= \dfrac{\Delta C_{net}}{C_{nom}} = 0.0925$