4F1 2006 Crib

1 (a) (i) The two degrees of freedom refer to the *feedback* part of the design, which is essentially the design of the return ratio $L(s)$ which combines $F(s)$ and $K(s)$ together, and the *feedforward* part which is the design of the transfer function relating $y(t)$ to $r(t)$ and amounts to the design of $H(s)$ as the last step. [10%]

(ii) Assuming that the return-ratio has already been designed, it is possible to achieve any transfer function $R(s)$ relating $y(t)$ to $r(t)$ which contains the right half plane zeros of $G(s)$ and which rolls off at high frequency at least as fast as $G(s)$. [10%]

(b) (i)

$$\begin{aligned} E(s) &= \frac{1}{F(s)K(s)/s^2+1}\cdot\frac{1}{s} \\ &= \frac{s}{F(s)K(s)+s^2} \end{aligned}$$

which must have a zero at the origin since $F(0)K(0) \neq 0$ for any stabilising control scheme. [10%]

(ii) By definition

$$E(s) = \int_0^\infty e(t)e^{-st}\,dt.$$

Setting $s=0$ gives the required result.

(iii) The equation

$$\int_0^\infty e(t)\,dt = 0$$

means that $e(t) = 1-y(t)$ cannot be positive for all $t \geq 0$ which means that $y(t)$ must exceed unity at some time. [10%]

(c) (i) Since $G(s)$ has no right half plane zeros and has second-order roll off at high frequency, all that is required of $R(s)$ is that it is stable and has second-order roll off at high frequency, which it does. We can now check that

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}$$

which means that $y(t) = 1-e^{-t}-te^{-t}$ in response to a step input, which never exceeds unity. [20%]

(ii) There are many possible solutions. E.g. choose

$$K(s) = \frac{s+1}{s+10}, F(s) = 1$$

which can be shown to be stabilising from the Bode diagram or the Routh-Hurwitz criterion. This gives

$$\frac{GK}{1+GKF} = \frac{s+1}{s^3+10s^2+s+1}$$

and the design can be completed with the choice of

$$H(s) = \frac{s^3+10s^2+s+1}{(s+1)^3}.$$ [25%]

Examiner's comment: This was a fairly typical two-degree-of-freedom design problem in which the candidate is led through the sequence of design steps. There were a number of good solutions but a number of candidates failed to connect the basic theory expressed in the bookwork at the start of the question to the information derived in the intermediate question parts, which then compromised the final design.

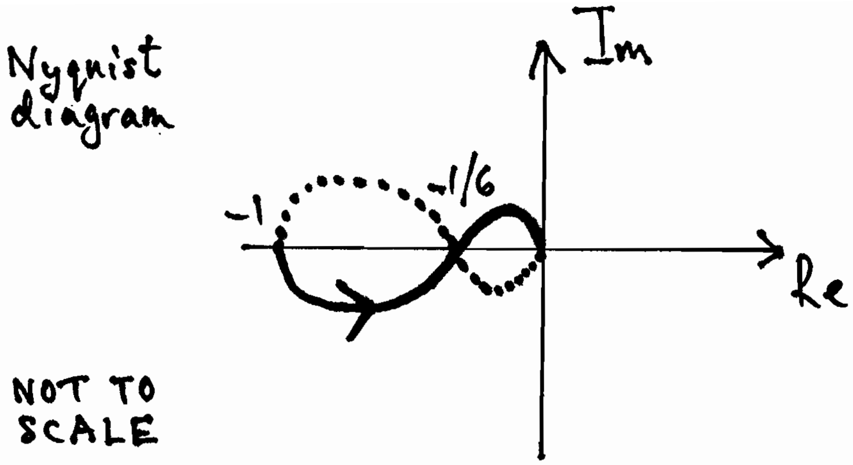
2 (a) (i) Straight line approximations on the Bode diagram give slopes of 0 at low frequency -20dB/dec at high frequency and around -30dB/dec at 10 rad/sec, which correspond to phases of $0°$, $-90°$ and $-135°$ respectively. The excess phase starts and ends at $180°$ and rises by the more than $90°$ in the mid-frequency range. [15%]

(ii) The fact that $G(s)$ has precisely one zero in the right half plane, which would contribute a phase change of $-180°$, means there must be precisely one pole in the right half plane to provide the net phase change of zero. The fact that the phase excess rises first means that the pole must occur at a lower frequency than the zero. This suggests an all-pass factor of the form:

$$-\frac{(1+s)(10-s)}{(1-s)(10+s)}$$

with a right half plane pole around 1 rad/sec and a right half plane zero around 10 rad/sec. [The actual transfer function used in this question was: $G(s) = -0.5(s-20)/(s-2)(s+5)$.] [20%]

(iii) This suggests that it would be difficult to achieve a gain crossover frequency much less than 1 rad/sec or much greater than 10 rad/sec. [10%]



(b) The sketch of the Nyquist diagram shows two crossings of the negative real axis which correspond to the 180° phase crossings on the Bode diagram, with corresponding magnitudes 0dB and -15dB approximately. This gives crossing points of -1 and -1/5.62. [The actual crossings are -1 and -1/6.] To achieve closed-loop stability we need one counterclockwise encirclement of the $-1/k$ point. We have:

$-\infty < -1/k < -1$, no encirclement

$-1 < -1/k < -1/6$, one counterclockwise encirclement

$-1/6 < -1/k < 0$, one clockwise encirclement

$0 < -1/k < \infty$, no encirclement.

This gives:

$-\infty < k < 1$, one RHP pole

$1 < k < 6$, no RHP poles (closed-loop stable)

$6 < k < \infty$, two RHP poles.

[20%]

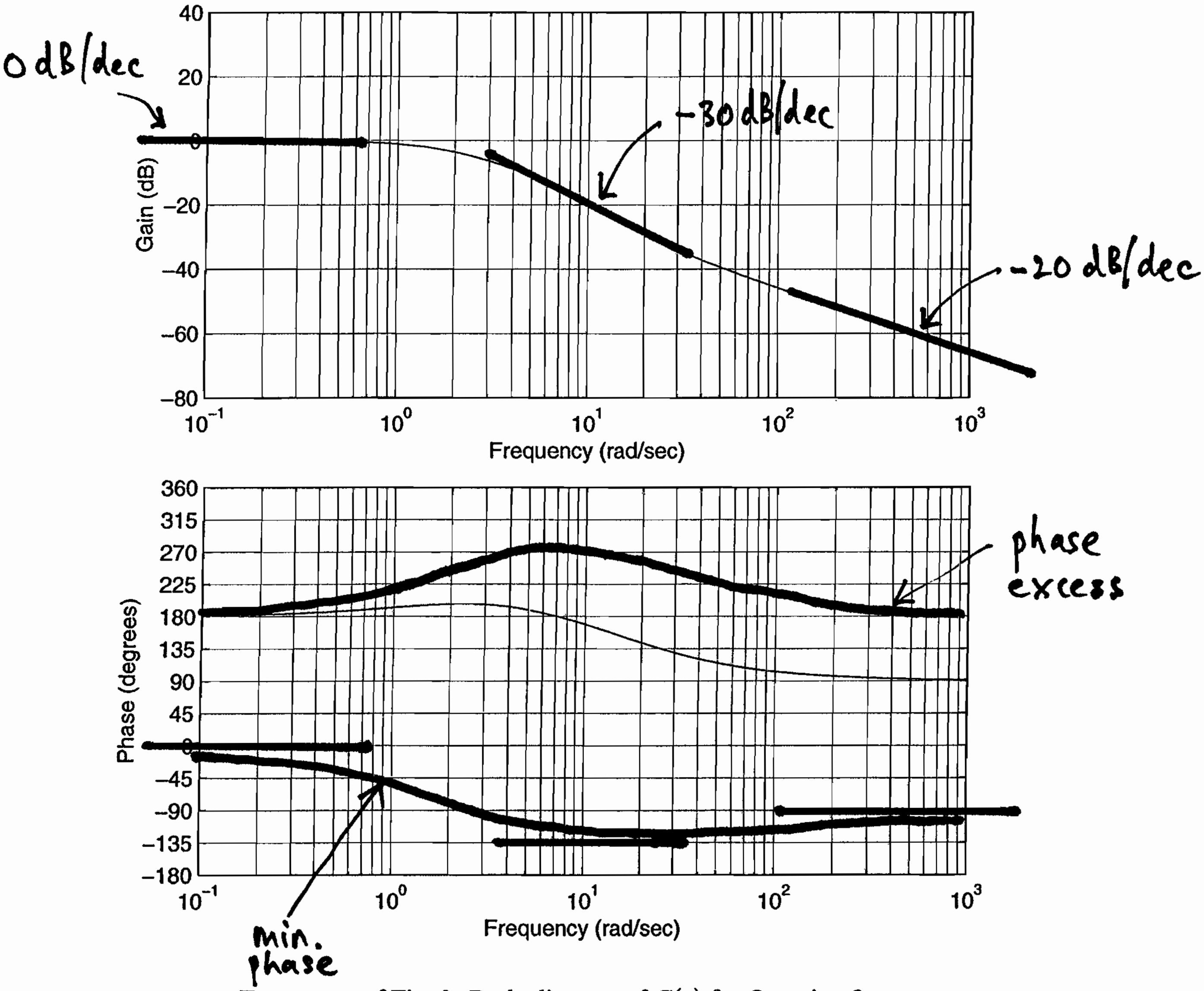
(c) Inspection of the Nyquist and Bode diagrams shows that the specifications can be achieved if phase advance amounting to about 55° is added at 10 rad/sec and at the same time the magnitude is increased by a factor of about 10 at the same frequency. This can be achieved by a compensator of the following form:

$$K(s) = 10\sqrt{10}\frac{s + 10/\sqrt{10}}{s + 10\sqrt{10}}.$$

[35%]

ENGINEERING TRIPOS PART IIB

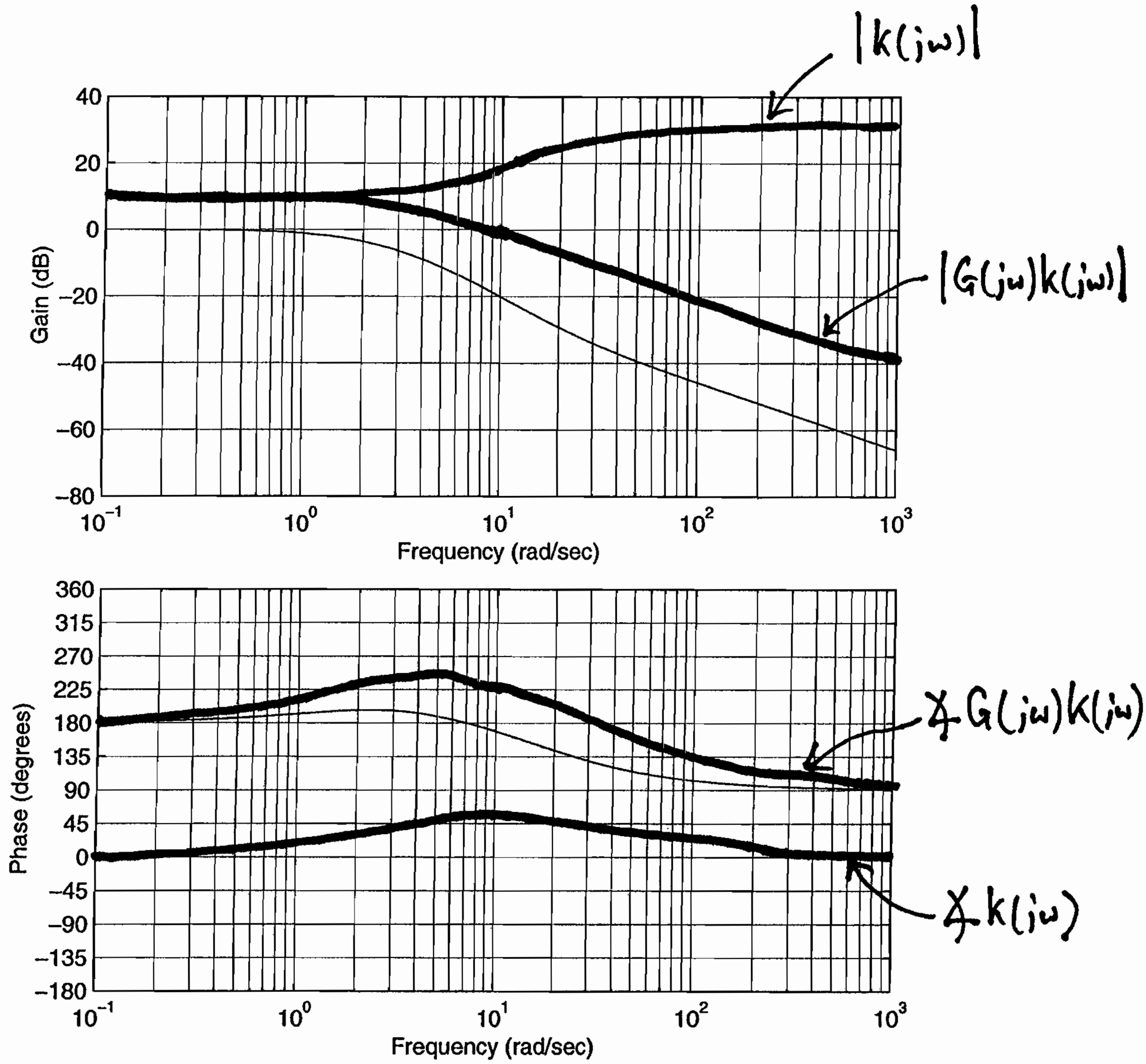
Wednesday 26 April 2006, Module 4F1, Question 2.



Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

ENGINEERING TRIPOS PART IIB

Wednesday 26 April 2006, Module 4F1, Question 2.



Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

Examiner's comment: This was a standard Bode gain-phase relationship and compensator design question. It was extremely pleasing that very many candidates obtained excellent solutions which demonstrated a very good understanding of the material.

3 (a) Since $\alpha = -1$ we have $S(1) = 1$. This gives

$$0 = \log|S(1)| = \frac{1}{\pi}\int_{\infty}^{\infty} \frac{1}{1+\omega^2} \log|S(j\omega)|d\omega.$$

Since $S+T=1$, specification C implies that: $|S(j\omega)| < 1.01$ for $\omega \geq 10$. Hence

$$\begin{aligned}
0 &= \int_0^{\infty} \frac{1}{1+\omega^2} \log|S(j\omega)|d\omega \\
&< \int_0^1 \frac{1}{1+\omega^2} \log\varepsilon\, d\omega + \int_1^{10} \frac{1}{1+\omega^2} \log 2\, d\omega + \int_{10}^{\infty} \frac{1}{1+\omega^2} \log 1.01\, d\omega \\
&= \left[\tan^{-1}\omega\right]_0^1 \log\varepsilon + \left[\tan^{-1}\omega\right]_1^{10} \log 2 + \left[\tan^{-1}\omega\right]_{10}^{\infty} \log 1.01 \\
&= 0.785 \log\varepsilon + 0.476.
\end{aligned}$$

Hence [50%]

$$\varepsilon > \exp(-0.476/0.785) = 0.545.$$

(b) (i) In the case that $\alpha = 1$, $G(s)$ is minimum phase with first-order roll off at high frequency. We can find a stabilising controller $K(s)$ such that

$$L(s) = G(s)K(s) = \frac{k}{s+1}$$

which gives

$$S(s) = \frac{s+1}{s+1+k}$$

which satisfies specification D for any $k > 0$. Also

$$|S(j)|^2 = \frac{2}{1+(1+k)^2}$$

which means that specification A will be satisfied providing [30%]

$$k > \left(\frac{2}{\varepsilon^2} - 1\right)^{1/2} - 1.$$

(ii) The strategy to achieve specification A requires high loop gain. For the nominal model assumed for $G(s)$ a high gain strategy is theoretically possible, however if there is any uncertainty at high frequency then this could lead to instability.

Another problem is caused by the factor $s^2 + 0.01s + 1$ in the plant numerator which implies a very small gain in $G(s)$ around the frequency 1 rad/sec. Thus to seek high loop gain could cause saturation of the plant actuators. [20%]

Examiner's comment: Part (a) was a very standard application of the Poisson integral to derive a lower bound on the sensitivity function. Most candidates knew what to do and carried out the integration accurately. Part (b) required candidates to observe that the plant was minimum phase with first-order role-off. Those who did so were mostly able to complete the question using a simple argument given in the lectures.

M. C. Smith
19 May 06