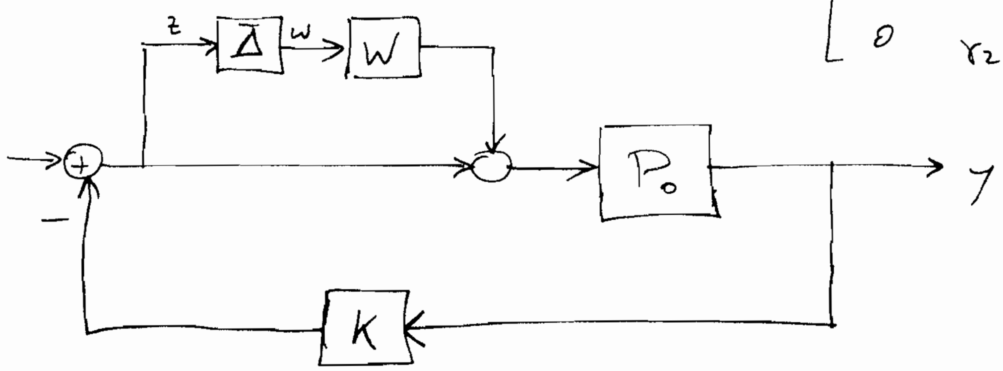
4F2 Dr G Vinnicombe gv@eng.cam.ac.uk

1. (a) $P(s) = P_o(s) + G(s)[\Delta_1, \Delta_2] = P_o(s) + G(s)\, 1^T \Delta$

where $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$

$= P_o(I + \Delta)$

$= P_o(I + W\bar{\Delta})$ where $W = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, $\|\bar{\Delta}_i\|_\infty < 1$



$y = P_o(Ww + z)$, $z = -Ky$

$= P_o(Ww - Ky) \iff y + P_o K y = P_o W w$

$\iff y = (I + P_o K)^{-1} P_o W w$

$\Rightarrow z = -K(I + P_o K)^{-1} P_o W w$

Let $M = T_{w \to z} = -K(I + P_o K)^{-1} P_o W$

Robust stability iff $\mu(M(j\omega)) \leq 1$ for all ω

(Note that $M = -(I + K P_o)^{-1} K P_o W$ also). [40%]

(b) · $\gamma = \max\{\gamma_1, \gamma_2\}$

$$\|(I+PK)^{-1}\|_\infty = \|(I+P_0(I+\Delta)K)^{-1}\|_\infty$$

$$= \sup_\omega \ \bar\sigma\left[(I+P_0K+P_0\Delta K)^{-1}\right]$$

$$= \sup_\omega \ \bar\sigma\left[(I+P_0K)^{-1}(I+P_0K)(I+P_0K+P_0\Delta K)^{-1}\right]$$

$$\leqslant \sup_\omega \ \|(I+P_0K)^{-1}\|_\infty \, \bar\sigma\left[(I+P_0K)(I+P_0K+P_0\Delta K)^{-1}\right]$$

Now $\bar\sigma\left[(I+P_0K)(I+P_0K+P_0\Delta K)^{-1}\right] = \dfrac{1}{\underline{\sigma}\left[(I+P_0K+P_0\Delta K)(I+P_0K)^{-1}\right]}$

$$= \frac{1}{\underline{\sigma}\left[I+P_0\Delta K(I+P_0K)^{-1}\right]}$$

$$\leqslant \frac{1}{1-\bar\sigma\left(P_0\Delta K(I+P_0K)^{-1}\right)}$$

$$= \frac{1}{1-\bar\sigma\left(\Delta K(I+P_0K)^{-1}P_0\right)}$$

$$\leqslant \frac{1}{1-\bar\sigma(\Delta)\,\bar\sigma\left(K(I+P_0K)^{-1}P_0\right)}$$

$$< \frac{1}{1-\gamma\,\bar\sigma\left(K(I+P_0K)^{-1}P_0\right)}$$

So, $f(\gamma) = \gamma\,\bar\sigma\left[K(I+P_0K)^{-1}P_0\right]$

[50%]

(c) By definition, the energy of output y

$$\|y\|_2^2 \leqslant \|(I+PK)^{-1}\|_\infty \cdot \|d\|_2^2 < \|(I+P_0K)^{-1}\|_\infty \frac{\|d\|_2^2}{1-f(\gamma)}$$

Hence, the inequality gives an upper bound on energy amplification from the input disturbance d to output y. This bound is smaller and smaller as $\gamma \to 0$, i.e., as the uncertainty gets smaller. [10%]

2) a) $$-\frac{\partial V}{\partial t} = \min_u \Big\{ \underbrace{x' Q x + u' R u}_{c(x,u)} + \underbrace{\frac{\partial V}{\partial x}(Ax+Bu)}_{\frac{\partial V}{\partial x}\frac{dx}{dt}} \Big\}$$

let $V = x' X x$

then $$\frac{\partial V}{\partial t} = \cancel{\quad} x' \dot{X} x + \cancel{\quad}$$

$$\frac{\partial V}{\partial x} = 2 x' X \cancel{\quad}$$

$$\Rightarrow x' \dot{X} x = \min_u \left\{ \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q + XA + A'X & XB \\ B^T X & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\}$$

$$= x' \left(Q + XA + A'X - XBR^{-1}B^T X \right) x$$

$$\Rightarrow \dot{X} = Q + AX + A'X - XBR^{-1}B^T \cancel{x} X$$

with b.c. $X(T) = X_T$ (50%)

b) $T = 1$, $R = \cancel{\quad} 1$, $Q = 0$

$$\cancel{\dot{x}} \quad \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$\Rightarrow \dot{X} = \cancel{A} XA + A'X - XBR^{-1}B^T X$$

$$\dot{X}^{-1} = AX^{-1} + X^{-1}A' - BR^{-1}B^T$$

let $X^{-1} = \cancel{\quad} \begin{matrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{matrix}$: $\begin{matrix} 0 & 0 \\ y_{11} & y_{12} \end{matrix} + \begin{matrix} 0 & y_{11} \\ 0 & y_{12} \end{matrix} - \begin{matrix} \cancel{\quad} 1 & 0 \\ 0 & 0 \end{matrix}$

$$\dot{y}_{11} = -1$$

$$\dot{y}_{12} = y_{11}$$

$$\dot{y}_{22} = 2 y_{12}$$

$\cancel{X(T)} = X(T) = \begin{matrix} \cancel{\gamma} & 0 \\ 0 & \gamma \end{matrix}$ as $\gamma \to \infty$

$\Rightarrow Y(T) = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$

$\cancel{y_{11} = 1}$ $y_{11} = \cancel{\quad} T - t$

$$\hat{y}_{12} = Tt - \frac{t^2}{2} \quad \cancel{a} \quad -\frac{T^2}{2}$$

$$y_{22} =$$

etc (not asking for equations)

then $X = Y^{-1}$

& $u(t) = -B^T X(t) \begin{pmatrix} \dot{x}(t) \\ x(t) \end{pmatrix}$

cost ϕ = $\begin{bmatrix} \dot{x}(0) \\ x(0) \end{bmatrix}' X(0) \begin{bmatrix} \dot{x}(0) \\ x(0) \end{bmatrix}$

50%

3) a) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace } G^{*}(j\omega)\, G(j\omega) \; d\omega$$

where $G^{*}(j\omega) = G'(-j\omega)$

if G is stable, ie $\cancel{Re}\ \lambda_i(A) < 0 \ \forall i$

20%

b) $\sup_t \ y'y \leq \|G\|_2^2 \ \|u\|_2^2$

In the SISO case

$$\max_t y(t) \leq \|G\|_2^2 \ \|u\|_2^2$$ 20%

from $$y(t) = \int_0^{t} g(\tau)\, u(t-\tau)\, d\tau$$

etc

c) Bookwork 60%