4F3 Nonlinear and Predictive Control: 2006 Solutions

Dr J.M. Maciejowski

May 18, 2006

1. (a) Since Φ has 2 columns, the number of states is $n = 2$.
   The number of elements in X is the same as the number of rows of Φ, which is 6. But each x_i in X has 2 elements, so the prediction horizon is $\boxed{N = 3}$ $(= 6/2)$.

   (b) Predictions are given by

$$\begin{aligned} x_1 &= Ax_0 + Bu_0 && (1)\\ x_2 &= A(Ax_0 + Bu_0) + Bu_1 = A^2x_0 + ABu_0 + Bu_1 && (2)\\ x_3 &= A(A^2x_0 + ABu_0 + Bu_1) + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2 && (3)\end{aligned}$$

   which can be collected together as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ A^3 \end{bmatrix} x_0 + \begin{bmatrix} B & 0 & 0 \\ AB & B & 0 \\ A^2B & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \qquad (4)$$

   Hence, from the given Φ and Γ,

$$\underline{A^3 = \begin{bmatrix} 21 & 34 \\ 34 & 55 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 13 \\ 21 \end{bmatrix}, \quad AB = \begin{bmatrix} 3 \\ 5 \end{bmatrix}} \qquad (5)$$

   (c) $x_1 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is equivalent to $Ax_0 + Bu_0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
   From part 1b and the given data we have $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So the constraint $x_1 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is equivalent to

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_0 &\leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} - Ax_0 \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} x(k) \end{aligned}$$

   (since $x_0 = x(k)$) which is equivalent to the constraint given in the question.

   (d) $-1 \leq \Delta u_s \leq 2 \Rightarrow -1 \leq u_s - u_{s-1} \leq 2$ which is equivalent to

$$\begin{aligned} u_s - u_{s-1} &\leq 2 \\ -u_s + u_{s-1} &\leq 1 \end{aligned}$$

Hence

$$\begin{aligned} u_0 - u_{-1} &\leq 2 \\ -u_0 + u_{-1} &\leq 1 \\ u_1 - u_0 &\leq 2 \\ -u_1 + u_0 &\leq 1 \\ u_2 - u_1 &\leq 2 \\ -u_2 + u_1 &\leq 1 \end{aligned}$$

which is the same as

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} U \leq \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{-1} \tag{6}$$

This corresponds to the top 6 rows of the given constraints, so by comparing corresponding elements we get $\boxed{a = 2,\ b = 1,\ c = -1,\ d = 1}$.

We have not yet considered the constraint $[1,\ 0]x_N \leq 2$. Since $N = 3$, and using results from part 1b, this is the same as

$$[1,\ 0](A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2) \leq 2 \tag{7}$$

or (again using results from part 1b)

$$[21,\ 34]x_0 + 13u_0 + 3u_1 + u_2 \leq 2 \tag{8}$$

Comparing this to the 7th row of the given constraints, and using the fact that we already know $a = 2$, we see that $\boxed{e = 13,\ f = 3,\ g = -21,\ h = -34}$.

2. (a) The assumption that d is constant can be expressed as $d(k+1) = d(k)$. Hence the given system can be expressed as

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} &= \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \end{aligned}$$

(b) We are told that A is stable. Since $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$ is block-diagonal, its eigenvalues are those of A and those of I, which are all at 1 (repeated m times, where m is the dimension of d — same as dimension of u). But we are told that A is stable, so the only eigenvalues on or outside the unit circle are located at 1, ie $\Lambda = \{1\}$. So the detectability criterion can be applied to the augmented system with $\lambda = 1$.

So, using the detectability criterion given in the question, the augmented system is detectable if and only if

$$\begin{bmatrix} I - \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} C & 0 \end{bmatrix} \end{bmatrix}$$

has full column-rank. But we can write this matrix as

$$\begin{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} C & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I-A & -B \\ 0 & 0 \\ C & 0 \end{bmatrix}$$

The middle row of zeros in this matrix does not affect the column-rank, so the detectability criterion becomes that the matrix $\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}$ should have full column-rank.

(c) i. $J(d,r) = 0 \Rightarrow y = r \Rightarrow Cx = r$. But from the constraint $x = Ax + Bu + Bd$ we have $(I-A)x - Bu = Bd$. Putting these together gives

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} Bd \\ r \end{bmatrix}$$

Now a standard result of linear algebra (also easy to prove) is that if M has full column-rank, then *if* $Mx = b$ has a solution then it is unique. In this case we know that there is a solution for x and u, since we are told that $J = 0$. Hence this equation has a unique solution if the coefficient matrix $\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}$ has full column-rank.

ii. As above, $J(d,r) = 0 \Rightarrow y = r \Rightarrow Cx = r$. But now we have $C = 1$, so $x = r$. Also we have the constraint $x = Ax + Bu + Bd$, with $A = 0.5$ and $B = 1$, so we have

$$\begin{aligned} x &= 0.5x + u + d \\ \Rightarrow 0.5x &= u + d \\ \Rightarrow u &= 0.5r - 2 \quad (\text{since } x = r,\ d = 2) \end{aligned}$$

But we have

$$\begin{aligned} &-3 \le \ u + d \ \le 3 \\ \text{and} \quad &-3 \le \ u \ \le 3 \end{aligned}$$

so substituting for u and d we have

$$\begin{aligned} &-3 \le \ 0.5r \ \le 3 \\ \text{and} \quad &-1 \le \ 0.5r \ \le 5 \end{aligned}$$

which are both satisfied if $-2 \le r \le 6$. So $\boxed{r < -2 \text{ or } r > 6} \Rightarrow J(d,r) > 0$.

3. (a) Let $u = f(e)$. Then the describing function is defined as $N(E) = \frac{U+jV}{E}$, where

$$U + jV = \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)(\sin\theta + j\cos\theta)d\theta$$

Since $f(e)$ is an odd function, $V = 0$. So only U needs to be calculated:

$$\begin{aligned} U &= \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta d\theta \\ &= \frac{4}{\pi}\int_0^{\pi/2} f(E\sin\theta)\sin\theta d\theta \\ &= \frac{4}{\pi}\int_0^{\sin^{-1}(d/E)} 0 d\theta + \frac{4}{\pi}\int_{\sin^{-1}(d/E)}^{\pi/2} \sin\theta d\theta \\ &= \frac{4}{\pi}[-\cos\theta]_{\sin^{-1}(d/E)}^{\pi/2} \\ &= \frac{4}{\pi}\cos\left[\sin^{-1}\left(\frac{d}{E}\right)\right] \\ &= \frac{4}{\pi}\sqrt{1-\left(\frac{d}{E}\right)^2} \end{aligned}$$

since $\cos(\sin^{-1}x) = \sqrt{1-x^2}$ (by elementary geometry). Therefore

$$N(E) = \frac{4}{\pi E}\sqrt{1-\left(\frac{d}{E}\right)^2}$$

(b) If $E < d$ then the output of the nonlinearity is always zero, and hence $\boxed{N(E) = 0}$.

(c) Differentiating $N(E)$, using the formula for the derivative of a quotient, gives

$$\begin{aligned} N'(E) &= \frac{2\{1-(d/E)\}^{-1/2}(2d^2E^{-3})\pi E - 4\pi\{1-(d/E)\}^{1/2}}{\pi^2 E^2} \\ &= \frac{4\pi d^2 E^{-2} - 4\pi\{1-(d/E)^2\}}{\pi^2 E^2\sqrt{1-\left(\frac{d}{E}\right)^2}} \end{aligned}$$

which is 0 when

$$\frac{d^2}{E^2} = 1 - \frac{d^2}{E^2} \Rightarrow \underline{E = d\sqrt{2}}$$

Sketching the graph:

We know that $N(E) = 0$ for $E < d$. Also from the expression, $N(E) \to 0$ as $E \to \infty$. This agrees with intuition, because the output approaches a square wave of amplitude 1 although the input becomes a sine wave of infinite amplitude, so the 'equivalent gain' becomes zero. When E becomes just bigger than d the output suddenly becomes non-zero, initially as narrow pulses, so one expects the 'equivalent gain' to increase to some maximum, then decrease towards 0. We already know from above that the maximum occurs for $E = d\sqrt{2}$. This maximum value is given by

$$E_{max} = E(d\sqrt{2}) = \frac{4}{\pi d\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{2}{\pi d}$$

So the graph looks like Fig. 1.

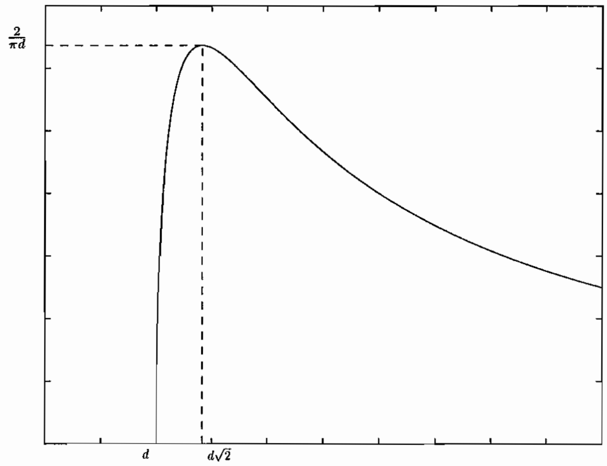

Figure 1: Graph of $N(E)$

(d) The describing function method predicts a limit-cycle oscillation when

$$G(j\omega) + \frac{1}{N(E)} = 0$$

ie when the graphs of $G(j\omega)$ and $-1/N(E)$ intersect. Since $N(E)$ is real and $N(E) \geq 0$, the graph of $-1/N(E)$ lies on the negative real axis. Since $0 \leq N(E) \leq 2/(\pi d)$ (from part 3c), we have $-\infty \leq -1/N(E) \leq -\pi d/2$. So an *absence* of limit-cycle oscillation will be predicted if the graph of $G(j\omega)$ crosses the real axis only at points to the right of $-\pi d/2$.

(*Note:* Every point on the graph of $-1/N(E)$ corresponds to two values of E, one with $N'(E) > 0$ and the other with $N'(E) < 0$. So each crossing would correspond to two limit-cycles, one stable and the other unstable. Therefore the absence of crossings is required.)

Now $G(j\omega) = k/[j\omega(j\omega+1)^2]$ and $k > 0$, so $\arg G(j\omega) = -\frac{\pi}{2} - 2\tan^{-1}(\omega)$ and hence $\arg G(j\omega)$ decreases monotonically from $-\pi/2$ to $-3\pi/2$ as ω increases from 0 to ∞. Thus it has only one intersection with the real axis, at some negative value. Let this intersection occur at frequency ω_0. Then

$$\arg G(j\omega_0) = -\frac{\pi}{2} - 2\tan^{-1}(\omega_0) = -\pi \Rightarrow \tan^{-1}(\omega_0) = \frac{\pi}{4} \Rightarrow \omega_0 = 1$$

Thus the crossing of the real axis occurs at

$$G(j1) = \frac{k}{j(j+1)^2} = -\frac{k}{2}$$

So an intersection of the two graphs does *not* occur if

$$-\frac{k}{2} > -\frac{\pi d}{2} \Rightarrow \underline{k < \pi d}$$

The situation is shown in Fig. 2.

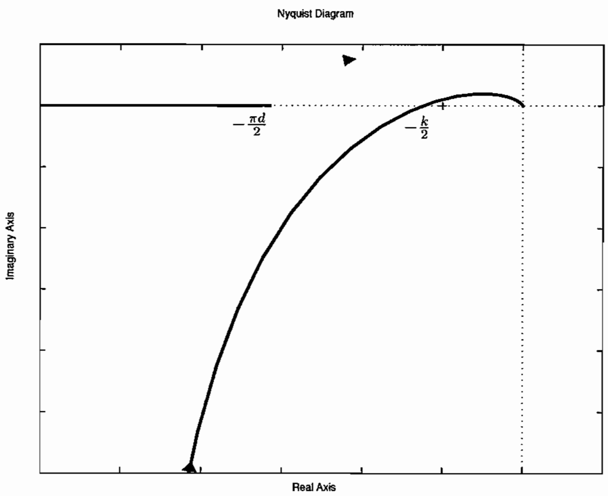

Figure 2: Graphs of $G(j\omega)$ and $-1/N(E)$

4. (a) A function $f(.)$ is Lipschitz continuous if

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for some constant $L > 0$.

Without assuming such a condition, we would not be sure that a solution of $\dot{x} = f(x)$ existed, or that it was unique. Both *existence* (for all time) and *uniqueness* of solutions are required for reasonable models of engineering systems, and are guaranteed by Lipschitz continuity. Another desirable consequence of Lipschitz continuity is *continuity of the solution with respect to initial conditions.*

(b) LaSalle's theorem states that if S is an invariant set in which $\dot{V}(x) \leq 0$, and M is the largest invariant set in S on which $\dot{V}(x) = 0$, then all trajectories which start in S approach M as $t \to \infty$.

When specialised to an equilibrium (assumed to be at 0), LaSalle's theorem states that if S is an invariant set in which $V(x) > 0$ and $\dot{V}(x) \leq 0$, and the set $\{x \in S | \dot{V}(x) = 0\}$ contains no trajectories other than $x(t) \equiv 0$, then 0 is asymptotically stable, and all trajectories starting in S converge to 0.

Lyapunov's method for establishing asymptotic stability of an equilibrium state requires that a continuous function $V(x)$ be found such that $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and $\dot{V}(x) < 0$ for all $x \neq 0$ in some neighbourhood of the equilibrium (assuming that a change of coordinates has been made so that the equilibrium is at 0). LaSalle's theorem extends this by:

- Relaxing the condition on $\dot{V}$ — which frequently does not hold, even when the equilibrium is asymptotically stable, and (sometimes) by removing the requirement that $V(x) > 0$ for $x \neq 0$.
- Furthermore, the general form of LaSalle's theorem allows convergence to invariant sets such as limit-cycles to be proved, not just to isolated equilibria.
- Also, finding an invariant set S in which the condition on $\dot{V}$ holds allows estimation of a region of attraction — with the Lyapunov method the estimate is always of the form $\{x | V(x) \leq c\}$.

(c) Clearly $(x_1 = 0, x_2 = 0) \Rightarrow (\dot{x}_1 = 0, \dot{x}_2 = 0)$, so (0,0) is an equilibrium state.

We have $V(0) = \int_0^0 h_1(y)dy + \frac{1}{2}0^2 = 0$.

For $x_1 \neq 0$ we have $\int_0^{x_1} h_1(y)dy > 0$, since $y < 0 \Rightarrow dy < 0$ and hence $h_1(y)dy > 0$ (as a consequence of $yh_1(y) > 0$). Hence we have $V(x) > 0$ for $x \neq 0$ and $x_1 < Y$.

$$
\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\
&= h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] \\
&= -x_2 h_2(x_2) \leq 0 \quad \text{if} \quad x \neq 0 \quad \text{and} \quad |x_1| < Y, |x_2| < Y.
\end{aligned}
$$

This proves that 0 is stable, by Lyapunov's stability theorem.

But we cannot assert that $\dot{V} < 0$ since $x_2 = 0$ may occur when $x \neq 0$. So try LaSalle's theorem:

Let $S = \{x | V(x) \leq V_0\}$, where V_0 is chosen such that this entire level set satisfies $|x_1| < Y$ and $|x_2| < Y$. S is invariant since $\dot{V}(x) \leq 0$ (shown above). Suppose that $x(t) \in S$ and $x(t) \neq 0$ but $\dot{V}(x(t)) = 0$. Then $x_2(t) = 0$, so $x_1(t) \neq 0$. Hence $h_1(x_1(t)) \neq 0$, since $yh_1(y) > 0$ for $y \neq 0$. So for small enough $\tau > 0$ we have $x_2(t+\tau) \approx \dot{x}_2(t)\tau = h_1(x_1(t))\tau \neq 0$ and hence $\dot{V}(x(t+\tau)) < 0$. Thus the only trajectory that can remain in the set $\{x \in S | \dot{V}(x) = 0\}$ is $x(t) \equiv 0$, and hence 0 is asymptotically stable, by LaSalle's theorem.

Note: Why does the question specify that h_1 and h_2 are Lipschitz continuous? Since we do not know what these functions are, without knowing that they are Lipschitz continuous the above argument could not even get started, since we would not even know whether the differential equations have a solution.