1) (a)(i) $(\underline{e}\times\underline{x})\cdot(\underline{e}\times\underline{x}) = \varepsilon_{ipq}\, e_p x_q \varepsilon_{ijk}\, e_j x_k$
$= (\delta_{pj}\delta_{qk} - \delta_{pq}\delta_{qj})\, e_p e_j x_q x_k$

$= e_p e_p x_q x_q - e_p e_q x_p x_q$
$= x_q x_q - e_p e_q x_p x_q$ as $|\underline{e}| = 1$

(ii) $\nabla\cdot\left[\dfrac{\underline{e}\times\underline{x}}{|\underline{e}\times\underline{x}|}\right] = \dfrac{\partial}{\partial x_i}\left[\varepsilon_{ijk}\, e_j x_k \left(x_q x_q - e_p e_q x_p x_q\right)^{-1/2}\right]$

$= \varepsilon_{ijk}\, e_j \Big[\delta_{ik}\left(x_q x_q - e_p e_q x_p x_q\right)^{-1/2}$

$- \frac{1}{2} x_k \left(x_r x_r - e_r e_s x_r x_s\right)^{-3/2}\left(2x_q\delta_{iq} - e_p e_q \delta_{ip} x_q - e_p e_q x_p \delta_{iq}\right)\Big]$

$= 0$ because every term contains a symmetric/antisymmetric combination.
Respectively $\varepsilon_{ijk}\delta_{ik}$, $\varepsilon_{ijk} x_k x_i$, $\varepsilon_{ijk} e_j e_i$ twice.

$\dfrac{\underline{e}\times\underline{x}}{|\underline{e}\times\underline{x}|}$ is the unit vector in the azimuthal direction for spherical polar coordinates: "$\underline{e}_\phi$"

(b) $I = \iiint_V \nabla\phi\cdot\nabla\phi \, dV = \iiint_V \dfrac{\partial\phi}{\partial x_i}\dfrac{\partial\phi}{\partial x_i}\, dV$

So if $\phi \to \phi + \delta\phi$, $I + \delta I \simeq \iiint_V \left[\dfrac{\partial\phi}{\partial x_i}\dfrac{\partial\phi}{\partial x_i} + 2\dfrac{\partial\phi}{\partial x_i}\dfrac{\partial\delta\phi}{\partial x_i}\right] dV$

$\therefore\ \delta I = 2\iiint_V \dfrac{\partial\phi}{\partial x_i}\dfrac{\partial\delta\phi}{\partial x_i}$

Integrate by parts using Stokes' theorem:

$$\delta I = 2\left[-\iiint_V \frac{\partial^2\phi}{\partial x_i\partial x_i}\,\delta\phi\, dV + \iint_C \frac{\partial\phi}{\partial x_i}\,\delta\phi\, n_i\, dS\right]$$

So for $\delta\Phi = 0$ for every possible $\delta\phi$, require both integrals to be zero.
From the volume integral, by the usual argument, need $\dfrac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$, i.e. $\nabla^2 \phi = 0$

From the surface integral, if $\delta\phi$ is arbitrary we require $\dfrac{\partial \phi}{\partial x_i} n_i = 0$ on S,

ie $\underline{n} . \nabla \phi = 0$, where $\underline{n}$ is the outward-pointing normal to S.

2) (a) For small increments $\delta\theta, \delta\phi$, the distances moved on the surface of the sphere are $a\delta\theta$, $a\sin\theta\,\delta\phi$ respectively. These two movements are in orthogonal directions, so the total distance δl satisfies

$$\delta l^2 = a^2\delta\theta^2 + a^2\sin^2\theta\,\delta\phi^2$$
$$= a^2\left(\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right)\delta\phi^2$$

∴ Total path length $L = a^2\int_{\phi_1}^{\phi_2}\left(f'^2 + \sin^2 f\right)^{1/2} d\phi$

if $\theta = f(\phi)$.

a^2 is constant, so need to minimise the given integral.

(b) Euler-Lagrange equation is

$$\cancel{\frac{d}{d\phi}\left(2f'\right) - 2\sin f\cos f}$$

$$\frac{d}{d\phi}\left[\frac{1}{2}\left(f'^2+\sin^2 f\right)^{-1/2}\cdot 2f'\right] - \frac{1}{2}\left(f'^2+\sin^2 f\right)^{-1/2}\cdot 2\sin f\cos f = 0$$

$$\therefore\ f''\left(f'^2+\sin^2 f\right)^{-1/2} - \frac{1}{2}\left(f'^2+\sin^2 f\right)^{-3/2}\left(2f'f''+2\sin f\cos f\right) - \left(f'^2+\sin^2 f\right)^{-1/2}\sin f\cos f = 0$$

$$\therefore\ f'' - \sin f\cos f - \frac{f'f'' + \sin f\cos f}{f'^2 + \sin^2 f} = 0$$

$f = $ constant means $f' = 0$, $f'' = 0$

∴ require $\sin f\cos f = 0$

So either $\sin f = 0$, ie $\theta = 0$ or π, ie a single

point at the "N pole" or "S pole"
<u>or</u> $\cos f = 0$ i.e. $\theta = \pi/2$, i.e. the path lies around the equator, which is a great circle.

(c) We are free to choose the axis about which we define our polar angles θ, ϕ.

For any given pair of end points for the path, we can always choose our axis so that both points lie on the equator, at $\theta = \pi/2$.

Then (b) shows that a possible path which satisfies the minimum length condition is the one which follows round the equator, keeping $\theta = \pi/2$. Could go either way – each is a <u>local</u> minimum, but the shorter of the two is the global minimum length (or geodesic).

3). a). $a_{11}=1$, $a_{12}=-\frac{3}{2}$, $a_{22}=2$

$$a_{12}^2 - a_{11}a_{22} = \frac{9}{4} - 2 = \frac{1}{4} > 0$$

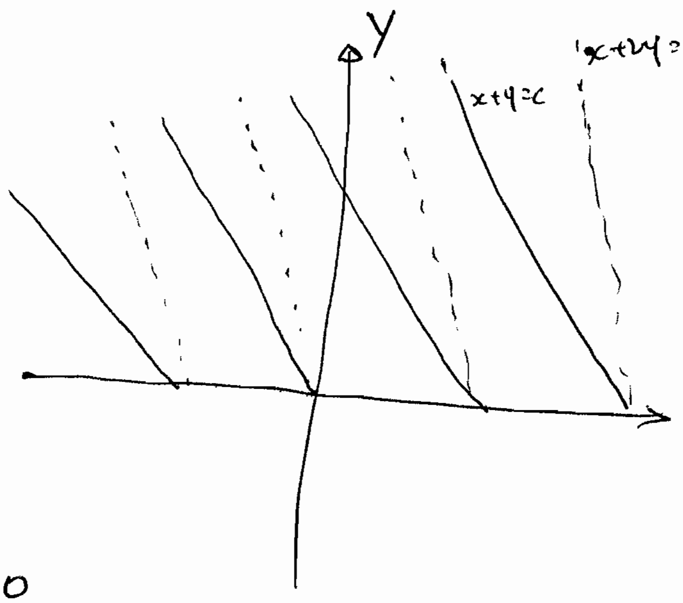
Hyperbolic equation

b). Characteristics equation.

$$\frac{dy}{dx} = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4}-2}}{1} = \begin{cases} -2 \\ -1 \end{cases}$$

$$\left.\frac{dy}{dx}\right|_1 = -2. \qquad y + 2x = C_1$$

$$\left.\frac{dy}{dx}\right|_2 = -1 \qquad y + x = C_2$$



c). let $\xi = y+2x$, $\eta = y + x$ $\quad > \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

further let $\alpha = \xi - \eta = x$, $\beta = \xi + x = 2y + 3x$ $\quad \Rightarrow \quad \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} = 0$

general solution: $u = f(\xi) + g(\eta)$

$$\Rightarrow u(x,y) = f(x+y) + g(2x+y)$$

3) [illegible]

d). boundary conditions.

$u(x,0)=1. \quad \frac{\partial u}{\partial y}(x,0)=1 \quad , \quad x \in (-\infty,\infty)$

$f(x)+g(2x)=1.$

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial f}+2\frac{\partial u}{\partial g}=1. \quad \Rightarrow$ by inspection

f and g must be linear function of ξ and η.

$f(\xi)=1-\xi \qquad g(\eta)=2\eta+1$

$f+g=u=2\eta+1-\xi=2y+2x+1-y-2x=y+1$

$\therefore \underline{\underline{u(x,y)=y+1}} \quad , \quad u(x,0)=1 \quad \frac{\partial u}{\partial y}=\frac{du}{dy}=1$

satisfy both conditions.

e). If u is only defined at $x[-1,1]$ and $y=0$.

$u(-1,2)=2+1=3$ contribution from $f=0$.
contribution from $g=3$.

$u(1,2)$ undefined as it lies outside of the domain of influence of boundary condition.

4).

a). Boundary condition:

$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ as no heat flux through $x=0$

or. $T_x(0,t) = 0$.

b). Green's function for the problem can be written as.

$$\frac{\partial G}{\partial t} - k\frac{\partial^2 G}{\partial x^2} = \delta(x-x_0)\cdot\delta(t-t_0)$$

$$G(x,0) = 0$$

This can be rewritten as:

$$\frac{\partial G}{\partial t} - k\frac{\partial^2 G}{\partial x^2} = 0 \;; \quad G(x,t;x_0,t_0)\Big|_{t=t_0} = \delta(x-x_0)$$

~~$G(x,t_0) = \delta(x-x_0)$~~

$x_0 \in (0,\infty)$. Because of $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$. T is symmetrical about $x=0$,

the solution can be extended to $x_0 < 0$, with $f(x_0) \equiv 0$.

from notes (P8 or P13)

$$G(x,t;x_0,t_0) = \int_0^\infty \delta(x-x_0)\frac{1}{\sqrt{4k\pi(t-t_0)}}e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}\,dx_0$$

by shifting $t=0$ to $t=t_0$, integrate and use $\int \delta(x-x_0)f(x)dx = f(x_0)$

$$G(x,t;x_0,t_0) = \frac{1}{\sqrt{4k\pi(t-t_0)}}e^{\frac{-(x-x_0)^2}{4k(t-t_0)}}$$ for $x>0$, $t>0$; $x_0>0$; $t_0>0$

the Green's function can be solved using separation of variable method or integral transformation method.

c). Let $\mathcal{L} \equiv \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$; $\quad \mathcal{L}T = f(x)$ and $\mathcal{L}G = \delta(x-x_0)\cdot\delta(t-t_0)$

$$\mathcal{L}\int_0^\infty\int_0^\infty G\cdot f\,dx_0\,dt_0 = \int_0^\infty\int_0^\infty \delta(x-x_0)\,\delta(t-t_0)f(x_0)\,dx_0\,dt_0 = f(x) = \mathcal{L}T$$

$$T(x,t) = \int_0^\infty\int_0^\infty G(x,t;x_0,t_0)\,f(x_0)\cdot dx_0\,dt_0$$

d). If both T_1 and T_2 are solutions of the problem, $\tilde{T} = T_1 - T_2$ must also ~~satisf~~ satisfy: $\frac{\partial \tilde{T}}{\partial t} - k\frac{\partial^2\tilde{T}}{\partial x^2} = 0$ and $\tilde{T}(x,0)=0$, $\tilde{T}_x(0,t)=0$

The solution can be written as that obtained in part (c) with $f(x)\equiv 0$, i.e.

$\tilde{T}(x,t) = \int\int^\infty G(x,t;x_0,t_0)\cdot 0\,dx_0\,dt_0 \equiv 0 \Rightarrow T_1 \equiv T_2$. $\therefore$ Solution is unique.