Final Crib 4M13 2006

SECTION A

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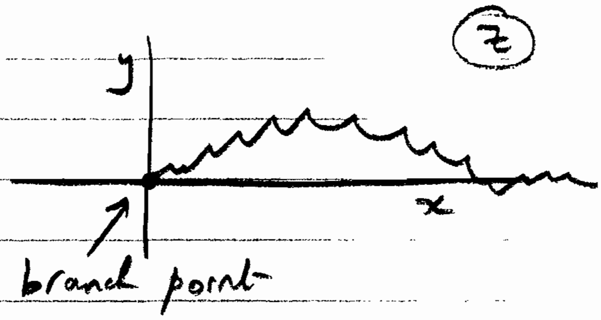
1. (a) A <u>branch point</u> and a <u>simple pole</u> are each singularities, about which no Taylor series exists

<u>Branch point</u>

e.g. $f(z) = z^{1/2}$

branch points at $z=0$ and $z=\infty$.



The function is multi-valued about a branch point.

e.g. write $z = re^{i\theta}$ Then $z^{1/2} = r^{1/2} e^{i\theta/2}$

A branch cut is arranged between 2 branch points.

A <u>simple pole</u> at $z = z_0$ behaves locally like $f(z) = \dfrac{A}{z-z_0}$. A is the residue at the pole

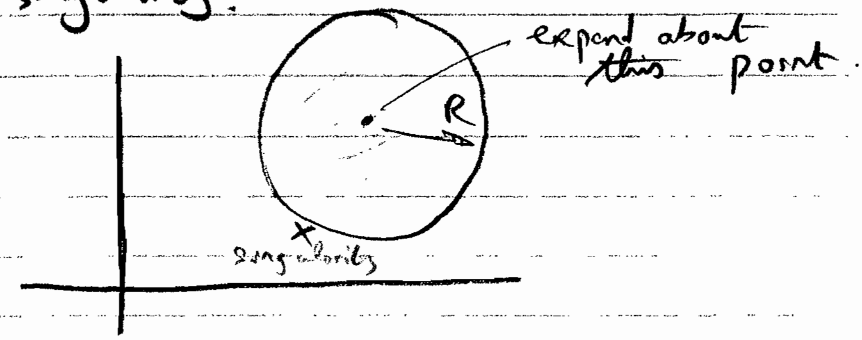
Example : $f(z) = \dfrac{3}{z-4} + \sin z$

The function can be expanded locally by a Laurent series expansion :

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

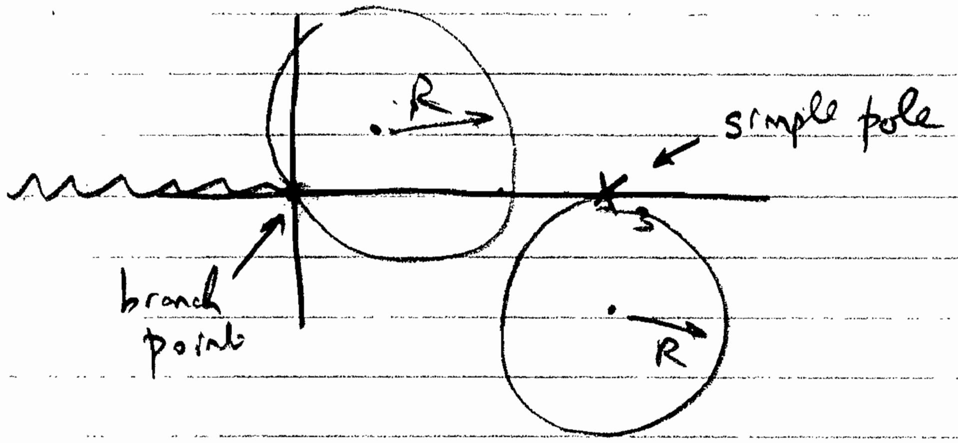
[25%]

(b) Expand a function about an ordinary point (a good point). The function fails to converge at the nearest singularity.



1. (b) contd.

Example : $f(z) = \dfrac{\sqrt{z}}{z-3}$



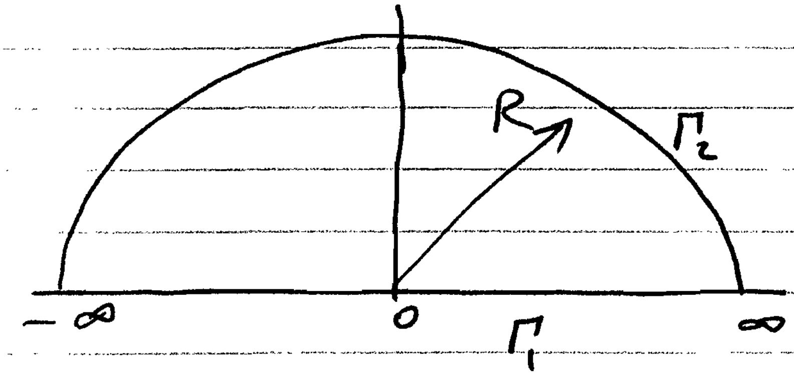
[15%]

A1. (a) $I = \int_0^\infty \frac{dx}{x^4 + a^4} = ?$

Note: integral exists and is well-behaved for $x \approx 0$ and for $x \approx \infty$.
Also the integrand is an <u>even</u> power of x, and so

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$$

Consider $J = \frac{1}{2} \oint_\Gamma \frac{dz}{z^4 + a^4}$



The contribution to the integral on Γ_2 is of order $\frac{1}{R^3}$ as $R \to \infty$, and so this contribution vanishes. Hence $J \equiv I$.

But we can use the residue theorem to evaluate J. In order to determine the singularities of the integrand, consider the denominator, $f(z) = z^4 + a^4$.

Zeros of $f(z)$ give poles of the integrand.

A1. (b) contd.

Now $f(z)$ has simple zeros at $z^4 + a^4 = 0$

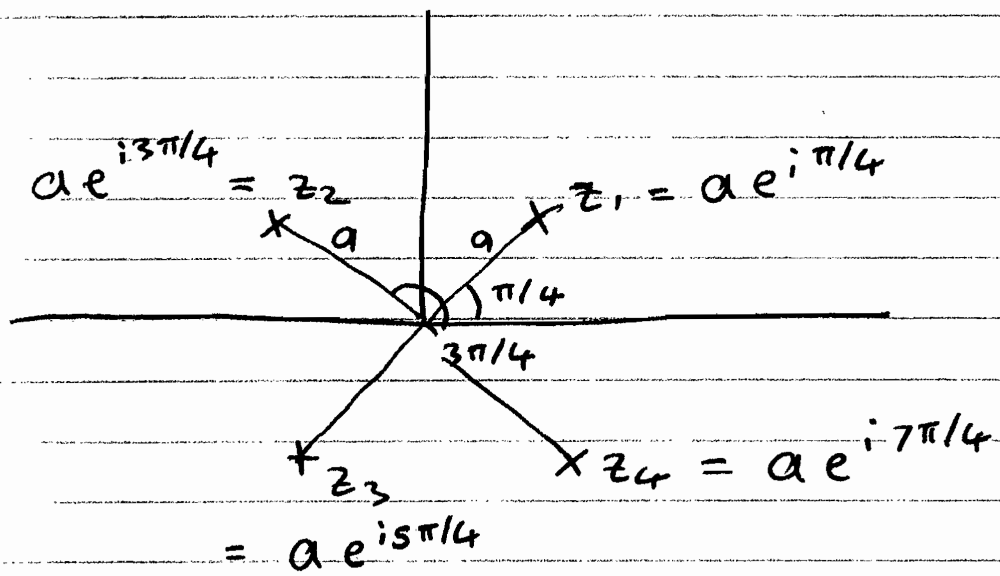
$$\Rightarrow \quad z^4 + a^4 e^{i2n\pi} = 0 \qquad n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad z^4 = a^4 e^{i(2n+1)\pi}$$

$$\Rightarrow \quad z = a e^{i\frac{(2n+1)}{4}\pi}$$

$$\Rightarrow \quad z = a e^{i\pi/4}, \quad a e^{i3\pi/4}, \quad a e^{i5\pi/4},$$

$$a e^{i7\pi/4}$$



The singularities at z_1 and z_2 are contained within Γ and their respective residues contribute to the integral.

Near $z = z_1$: $\quad f(z) \sim 0 + (z - z_1) f'(z_1) + \ldots$

$$f(z) = z^4 + a^4 \quad \Rightarrow \quad f'(z) = 4z^3$$

$$\Rightarrow \quad f(z) \sim (z - z_1) \cdot 4z_1^3 + \cdots$$

$\therefore$ Residue R_1 at $z = z_1$ is $2\pi i \cdot \frac{1}{2} \frac{1}{4z_1^3}$

$$z_1 = a e^{i\pi/4}$$

$$\Rightarrow \quad R_1 = \frac{\pi i}{4a^3} e^{-i3\pi/4}$$

Residue R_2 at $z = z_2 = a e^{i3\pi/4}$ is

$$R_2 = \frac{\pi i}{4a^3} e^{-i9\pi/4}$$

$$R_1 + R_2 = \frac{\pi i}{4a^3} \left(e^{-i3\pi/4} + e^{-i\pi/4} \right)$$

$$= \frac{\pi}{2\sqrt{2}a^3}$$

So $\quad I \equiv J = \dfrac{\pi}{2\sqrt{2}a^3}$

[60%]

2, (a) Taylor series for $f(z)$ about z_0 reads

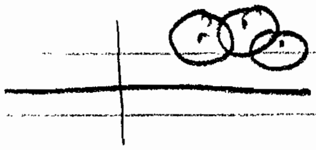
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \dots$$

Can expand about an <u>ordinary</u> point.

Use a Laurent series to expand about a pole of order N at $z = z_0$:

$$f(z) = \frac{a_{-N}}{(z - z_0)^N} + \frac{a_{1-N}}{(z - z_0)^{N-1}} + \dots a_0 + a_1(z - z_0) + \dots$$

The residue is a_{-1}.



Analytic continuation of $f(z)$ to cover the plane. [30%]

(b) (i) $$f(z) = \frac{\cancel{(z-4)}\, z^{1/3}}{\cancel{(z-4)}(z-2)}$$

So, simple pole at $z = 2$.
Branch cut at $z = 0$.

Residue at $z = 2$ is $2^{1/3}$. [20%]

(ii) $$f(z) = \frac{z \ln z}{\sin z}$$

At $z = 0$ $\quad f(z) \sim \dfrac{z \ln z}{z - \frac{1}{3!} z^3 + \dots} \sim \ln z$

So a branch point at $z = 0$.

At $z = \pm\pi, \pm 2\pi, \dots = \pm n\pi$ where

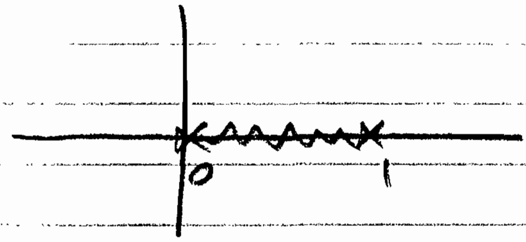
n is an integer, we have a simple pole.
but not $= 0$

Residue is $\dfrac{n\pi \ln n\pi}{(-1)^n}$

[30%]

(iii) $f(z) = z^{1/2}(z-1)^{1/2}$

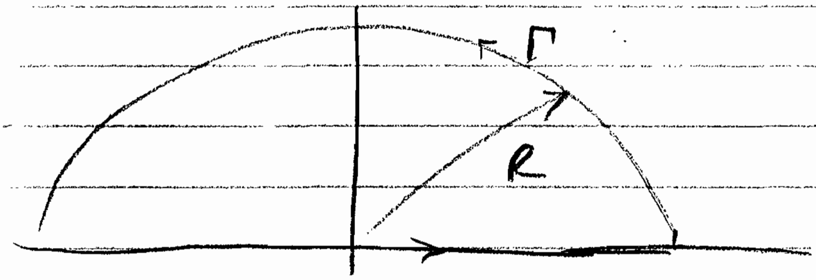
Branch points at $z = 0, 1$.



(b) Jordans lemma :

$$\lim_{R \to \infty} \int_\Gamma f(z)\, e^{iaz}\, dz \to 0 \quad \text{for } a > 0.$$

provided $f(z) \to 0$ at ∞.



Hence $\displaystyle\int_{-\infty}^{\infty} e^{iaz} f(z)\, dz = \oint e^{iaz} f(z)\, dz$

Useful for Fourier Transforms.

[20%]

Final crib

SECTION B

Q3

(a)
$$\frac{\partial f}{\partial V} = 1 - \frac{4\times10^5}{V^3 C}$$

$$\frac{\partial f}{\partial C} = 4\times10^3 - \frac{2\times10^5}{V^2 C^2}$$

$$\frac{\partial^2 f}{\partial V^2} = \frac{12\times10^5}{V^4 C}$$

$$\frac{\partial^2 f}{\partial V \partial C} = \frac{4\times10^5}{V^3 C^2}$$

$$\frac{\partial^2 f}{\partial C^2} = \frac{4\times10^5}{V^2 C^3}$$

For minimum $\underline{\nabla f} = 0$ and $\underline{\underline{\nabla^2 f}}$ is +ve definite

$$\frac{\partial f}{\partial V} = 0 \quad \Rightarrow \quad V^3 C = 4\times10^5 \quad (1)$$

$$\frac{\partial f}{\partial C} = 0 \quad \Rightarrow \quad V^2 C^2 = 50 \quad (2)$$

$$(1) \Rightarrow \quad V^6 C^2 = 16\times10^{10} \quad (3)$$

$$(3)\div(2) \Rightarrow \quad V^4 = 3.2\times10^9 \quad \Rightarrow \quad \underline{\underline{V = 237.8 \text{ kV}}}$$

$$\therefore (1) \Rightarrow \quad \underline{\underline{C = 0.02973 \text{ S}}}$$

For these values $\underline{\underline{\nabla^2 f}} = \begin{bmatrix} 0.01261 & 33.72 \\ 33.72 & 269185 \end{bmatrix}$

$|\nabla^2 f| = 2257$

The top-left entry of $\underline{\underline{\nabla^2 f}}$ is also > 0 (0.01261)

$\therefore$ $\underline{\nabla^2 f \text{ is +ve definite}}$

[25%]

(b) For $\underline{x}_0 = (100,\ 0.1)^T$

$$\underline{\nabla f}_0 = \begin{pmatrix} -3 \\ 2000 \end{pmatrix} \quad \Rightarrow \quad \underline{d}_0 = -\underline{\nabla f}_0 = \begin{pmatrix} 3 \\ -2000 \end{pmatrix}$$

Q3 (cont.)

$$\underline{\underline{H}}_0 = \nabla^2 f_0 = \begin{pmatrix} 0.12 & 40 \\ 40 & 40000 \end{pmatrix}$$

$$\alpha_0 = \underline{d}_0^T \underline{d}_0 / \underline{d}_0^T \underline{\underline{H}}_0 \underline{d}_0$$

$$\underline{d}_0^T \underline{d}_0 = 4.0 \times 10^6$$

$$\underline{d}_0^T \underline{\underline{H}}_0 \underline{d}_0 = (3 \quad -2000) \begin{pmatrix} 0.12 & 40 \\ 40 & 40000 \end{pmatrix} \begin{pmatrix} 3 \\ -2000 \end{pmatrix}$$

$$= (3 \quad -2000) \begin{pmatrix} -8 \times 10^4 \\ -8 \times 10^7 \end{pmatrix} = 1.6 \times 10^{11}$$

$$\therefore \alpha_0 = 2.5 \times 10^{-5}$$

$$\therefore \underline{x}_1 = \underline{x}_0 + \alpha_0 \underline{d}_0 = \begin{pmatrix} 100 \\ 0.1 \end{pmatrix} + 2.5 \times 10^{-5} \begin{pmatrix} 3 \\ -2000 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 100 \\ 0.05 \end{pmatrix}}}$$

$$\therefore \nabla f_1 = \begin{pmatrix} -7 \\ -4000 \end{pmatrix} \Rightarrow \underline{d}_1 = -\nabla f_1 = \begin{pmatrix} 7 \\ 4000 \end{pmatrix}$$

$$\underline{\underline{H}}_1 = \nabla^2 f_1 = \begin{pmatrix} 0.24 & 160 \\ 160 & 3.2 \times 10^5 \end{pmatrix}$$

$$\underline{d}_1^T \underline{d}_1 = 16 \times 10^6$$

$$\underline{d}_1^T \underline{\underline{H}}_1 \underline{d}_1 = (7 \quad 4000) \begin{pmatrix} 0.24 & 160 \\ 160 & 3.2 \times 10^5 \end{pmatrix} \begin{pmatrix} 7 \\ 4000 \end{pmatrix}$$

$$= (7 \quad 4000) \begin{pmatrix} 6.4 \times 10^5 \\ 1.28 \times 10^9 \end{pmatrix} = 5.12 \times 10^{12}$$

$$\therefore \alpha_1 = 3.125 \times 10^{-6}$$

$$\therefore \underline{x}_2 = \begin{pmatrix} 100 \\ 0.05 \end{pmatrix} + 3.125 \times 10^{-6} \begin{pmatrix} 7 \\ 4000 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 100 \\ 0.0625 \end{pmatrix}}}$$

[40%]

Q3 (cont).

(c) The Steepest Descent Method is clearly struggling to make adequate progress.
The convergence ratio for the SDM depends on the eigenvalues of the Hessian. Ideally the eigenvalues are large and close in value. In this case they are vastly different: for $\underline{H}_0$ they are 40000 and 0·08; for $\underline{H}_1$ they are $3{\cdot}2 \times 10^5$ and 0·16. These result in a very low convergence ratio and therefore very slow convergence.
There are also potential problems using the formula for α_k if the function is not well approximated by a quadratic. This can lead to overshooting of the minimum in the direction of search. This has happened here: $f_0 = 700$, $f_1 = 700$, $f_2 = 670$. These points are all on the same line ($V = 100$) so the search has overshot the line-search minimum on the first iteration. [20%]

(d) Newton's Method offers quadratic (rather than linear) convergence, but needs the Hessian to be computed and inverted, which can prove difficult for problems with many control variables. There can also be difficulties of convergence if the function is not well modelled as a quadratic.
The Conjugate Gradient Method needs the Hessian but not its inverse. It offers only linear convergence but with smaller convergence ratios than for the SDM. Restarting overcomes problems associated with the "quadraticness" of the objective function. [15%]

Q4.

(a)(i) Minimise $f = xy$

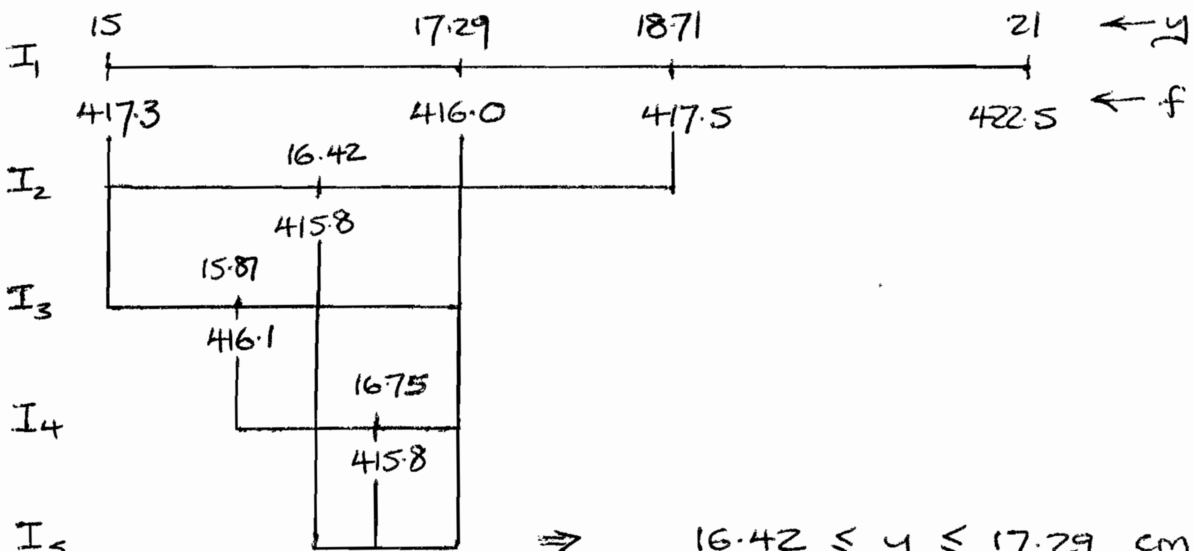
subject to $(x-6)(y-4) \geq 240$ [5%]

(ii) If the constraint is active

$$x = 6 + \frac{240}{y-4}$$

$$\therefore f = y\left(6 + \frac{240}{y-4}\right) = \frac{6y\,(y+36)}{(y-4)}$$ [10%]

(iii) For the GSLS method $\dfrac{\Delta y}{d} = 0.382$



$\Rightarrow$ $\underline{\underline{16.42 \leq y \leq 17.29}}$ cm

[35%]

(b) In standard form the problem is

Minimise $f = xy$

subject to $g_1 = 240 - (x-6)(y-4) \leq 0$

$\therefore L = f + \mu g_1 = xy + \mu\big(240 - (x-6)(y-4)\big)$

K-T conditions are

$\dfrac{\partial L}{\partial x}$: $\quad y + \mu(4-y) = 0 \qquad (1)$

$\dfrac{\partial L}{\partial y}$: $\quad x + \mu(6-x) = 0 \qquad (2)$

Q4 (cont.)

and $\quad \mu\left(240-(x-6)(y-4)\right) = 0 \qquad (3)$

If constraint is inactive $\mu = 0$

$\therefore$ (1) $\Rightarrow \quad y = 0$

(2) $\Rightarrow \quad x = 0$

But if $x = y = 0$ $\quad g_1$ is violated

$\therefore$ constraint must be active

[20%]

(c) If $\mu \neq 0$

(1) $\Rightarrow \quad y = \mu(y-4)$

(2) $\Rightarrow \quad x = \mu(x-6)$

$$\therefore \quad \frac{y}{x} = \frac{y-4}{x-6}$$

$$\therefore \quad \cancel{xy} - 6y = \cancel{xy} - 4x \quad \Rightarrow \quad x = 1.5y$$

If constraint is active

$$240 - (x-6)(y-4) = 0$$

$\therefore \quad 240 = (1.5y - 6)(y-4) \qquad$ as $x = 1.5y$

$$\therefore \quad 160 = (y-4)^2$$

$$\therefore \quad y = 4 + \sqrt{160} = \underline{\underline{16.649 \text{ cm}}}$$

$$\therefore \quad x = 1.5y = \underline{\underline{24.974 \text{ cm}}}$$

To confirm minimum need to check μ

From (1) $\quad \mu = y/(y-4) = \underline{\underline{1.316}}$

As $\mu > 0$ this <u>is</u> a minimum.

The GSLS has successfully identified the vicinity of the optimal value of y quite quickly. It is an efficient line search method.

[30%]