ENGINEERING TRIPOS PART IIB

Friday 5 May 2006 9 to 10.30

Module 4A12

TURBULENCE

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachments: 4A12 Data Card (3 pages)*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 (a) Explain the physical origin of the secondary flow in Kármán and Bodewadt layers. [30%]

(b) Use order of magnitude arguments to deduce that, for Kármán and Bodewadt layers,

$$\rho\frac{u_\theta^2}{r} \approx \rho\nu\frac{u_r}{\delta^2} \,.$$

Hence, or otherwise, show that the boundary layer thickness is of order $(\nu/\Omega)^{1/2}$, where $\Omega$ is the relative rotation between the fluid and the surface. You may assume that $u_r$ and $u_\theta$ are of the same order of magnitude in the layer. [35%]

(c) Sketch the secondary flow pattern in a stirred cup of tea. Explain how the tea comes to rest and estimate the spin-down time. [35%]

2 (a) State Kelvin's circulation theorem. [20%]

(b) Consider two adjacent points, A and B, on a dye line (a material line) in an inviscid fluid. Let $d\mathbf{l}$ be the displacement vector that links the two points, i.e. $d\mathbf{l} = \mathbf{x}_B - \mathbf{x}_A$. The change in $d\mathbf{l}$ in time $\delta t$ is $\delta(d\mathbf{l}) = (\mathbf{u}_B - \mathbf{u}_A)\delta t$ . Show that

$$\frac{D}{Dt}(d\mathbf{l}) = \big((d\mathbf{l})\cdot\nabla\big)\mathbf{u} \ .$$

Now, suppose that, at some initial instant, the dye is coincident with a vortex line, so that $d\mathbf{l} = \alpha\boldsymbol{\omega}(\mathbf{x}_A)$ at $t = 0$, $\alpha$ being some scalar. Use the inviscid vorticity equation to show that $D\alpha / Dt = 0$, and hence that

$$\frac{d\mathbf{l}(t)}{|d\mathbf{l}(t=0)|} = \frac{\boldsymbol{\omega}(t)}{|\boldsymbol{\omega}(t=0)|} \ .$$

Thus, the change in the direction and magnitude of $\boldsymbol{\omega}$ in a material element with time is the same as $d\mathbf{l}$. It follows that points A and B always lie on the vortex line, i.e. the vortex line moves like a dye line. What is the name associated with this law? [50%]

(c) The fact that vortex lines move like dye lines may also be obtained from Kelvin's theorem. Consider a thin, isolated vortex tube and a closed curve $C$ that encircles the vortex tube at $t = 0$. Use Kelvin's theorem to show that, if $C$ is a material curve always composed of the same fluid elements, then it must encircle the vortex tube for all times. Deduce that vortex lines move with the fluid. [30%]

3 Consider a 20 m x 20 m reception room with a ceiling height 8 m. The room contains on average one guest per square metre. The guests are moving about the room in random patterns with a characteristic velocity of 0.5 m $s^{-1}$, each of them doing work on the surrounding fluid at an average rate of 1 W.

(a) The mechanism that produces the velocity fluctuations in this problem is limited in the vertical direction up to the guests' height and in the two horizontal directions. Discuss qualitatively how the turbulent fluctuations become finite and approximately isotropic in the region between the guests' heads and the ceiling. [40%]

(b) Assuming that the turbulence is homogeneous and isotropic in the whole of the reception room and neglecting any wall effects, estimate the characteristic large-eddy lengthscale and timescale of the turbulence in the room. [40%]

(c) A 2 cm diameter neutrally-buoyant balloon is claimed by some guests to be small enough to respond to the full range of turbulent motions in the room. Is this correct? [20%]

4 Consider a turbulent axisymmetric jet issuing into stagnant surroundings from a nozzle of diameter $D$ with uniform velocity $U_0$. $x$ is the streamwise and $r$ the radial coordinate.

(a) Given that the axial velocity radial profile is self-similar and that the jet width $\delta$ increases linearly with $x$, show that the centreline jet velocity $U_c$ decreases as $x^{-1}$. [40%]

(b) Given additionally that the Reynolds stresses are self-similar, show that the eddy viscosity is independent of $x$. [40%]

(c) Show that the mass flow rate in the jet increases linearly with $x$. [20%]

**END OF PAPER**

# Cambridge University Engineering Department

# 4A12: Turbulence

# Data Card

Assume incompressible fluid with constant properties.

**Continuity:**

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

**Mean momentum:**

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

**Mean scalar:**

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

**Turbulent kinetic energy** ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

**Scalar fluctuations** ($\sigma^2 = \overline{\phi' \phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

**Energy dissipation:**

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

**Scalar dissipation:**

$$2\overline{N} = 2D\overline{\left(\frac{\partial\phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

**Scaling rule for shear flow, flow dominant in direction $x_1$:**

$$\frac{u}{L_{turb}} \sim \frac{\partial\overline{u}_1}{\partial x_2}$$

**Kolmogorov scales:**

$$\begin{aligned}\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4}\\ \tau_K &= (\nu/\varepsilon)^{1/2}\\ v_K &= (\nu\varepsilon)^{1/4}\end{aligned}$$

**Taylor microscale:**

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

**Eddy viscosity (general):**

$$\begin{aligned}\overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij}\\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial\overline{\phi}}{\partial x_j}\end{aligned}$$

**Eddy viscosity (for simple shear):**

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial\overline{u}_1}{\partial x_2}$$

# Vortex Dynamics Data Card

## Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

## Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint A \cdot d\boldsymbol{l}$

## Vector Identities

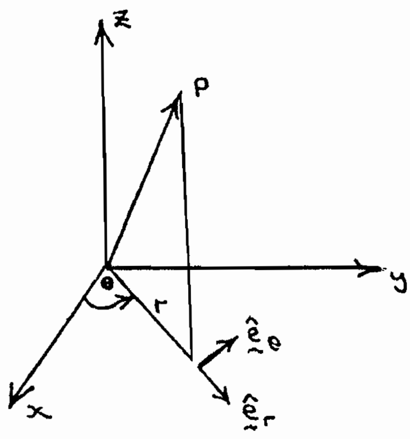
$\nabla(A \cdot B) = (A \cdot \nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$
$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$
$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times A) = 0$

## Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$




$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

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**4A12 – Turbulence**

**Numerical answers**

Q3. (b) Turbulent velocity = 0.5 m/s (given); Lengthscale = 1.25 m; Timescale = 2.5 s.
(c) Kolmogorov lengthscale = 0.4 mm