ENGINEERING TRIPOS PART IIB

Tuesday 25 April 2006 2.30 to 4

Module 4C2

DESIGNING WITH COMPOSITES

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Module 4C2 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A $[\pm 45]_S$ glass fibre-epoxy laminate is constructed from 1 mm thick plies of unidirectional Scotchply/1002 material, with material properties as given on the datasheet.

(a) Obtain the in-plane stiffness constants for the laminate. [45%]

(b) A sandwich beam is constructed with the glass laminate as face sheets, as illustrated in Fig. 1. Calculate the axial and effective bending stiffness of the beam, per unit width. The stiffness contribution from the foam core can be neglected. [30%]

(c) Discuss how such a beam might be constructed. [25%]

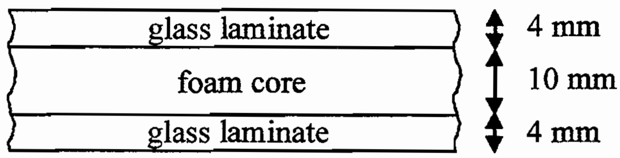


Fig. 1

2 (a) Describe how strain allowables can be used to design against failure of composite laminates. Explain the differences between the numerical values of strain allowable, for the various fibre composites and three loading directions quoted in Table 1 of the datasheet. [25%]

(b) For a $[\pm\theta]_S$ laminate, explain why the use of strain allowables may not give an accurate estimate of the failure load. Would you expect design using strain allowables to be conservative in these circumstances? [25%]

(c) Use strain allowables to estimate a suitable ply mix and total thickness for a CFRP laminate containing only 0°, ±45° and 90° plies, designed to carry in-plane line loads of $N_x = 2$ MN m^{-1} and $N_y = -1$ MN m^{-1} without failing. [25%]

(d) Without doing detailed laminate plate theory calculations, estimate the maximum axial and bending stiffnesses consistent with your beam design found in (c). [25%]

3 (a) Propose a physical model for the principal mode of failure of a multi-directional carbon fibre-epoxy laminate plate containing a hole under compression loading. Illustrate the dominant failure mechanism by a simple sketch. [35%]

(b) Propose a linear-softening crack tip cohesive zone law that accounts for the strain energy release rate G for compression cracking at the edge of the hole. What is this law capable of predicting? [20%]

(c) The critical crack closing displacement v_c for the micro-buckling of a carbon fibre composite at the edge of a hole in a plate under compression loading is given by

$$v_c = \frac{\pi d_f}{8}\left(\frac{V_f E_f}{2\tau_y}\right)^{\frac{1}{3}}$$

where d_f is the fibre diameter, V_f the fibre volume fraction, E_f the fibre Young's modulus and τ_y the in-plane shear yield strength of the laminate. Choosing appropriate values for the terms in the equation above (with justification) and, given that the theoretical un-notched compressive strength of the laminate is 740 MPa, estimate the toughness G_c of the laminate. [25%]

(d) Account for any discrepancy that is likely to be observed between the prediction of the **un-notched** compression strength of the laminate plate and that measured by experiment. [20%]

(TURN OVER

4 The CFRP wing panel of an aircraft carries an axial line load N_x and a shear line load N_{xy}. The panel is made of eight plies of AS/3501 carbon fibre epoxy material (material data on the data sheet) each of thickness 0.5 mm, and has a laminate lay-up of $[0_2, \pm 45]_S$. The x axis lies in the 0° direction. The $[Q]$ matrix for a unidirectional lamina of this material and the $[A]$ matrix for the laminate are as follows:

$$[Q] = \begin{bmatrix} 139 & 2.7 & 0 \\ 2.7 & 9.0 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{GPa}, \quad [A] = \begin{bmatrix} 368 & 68 & 0 \\ 68 & 109 & 0 \\ 0 & 0 & 85 \end{bmatrix} \text{GPa mm}$$

(a) For a fixed axial line load $N_x = 1.6$ MN m^{-1}, calculate the maximum shear load N_{xy} that the panel can carry for first ply failure. Use the maximum strain criterion and assume that failure is due to transverse tension in one of the 45° plies. [55%]

(b) For the case where both N_x and N_{xy} can vary, sketch the failure surface of the laminate in the $N_x - N_{xy}$ plane, for positive N_x, marking salient points and identifying expected failure modes. [45%]

END OF PAPER

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix}=[S]\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} \qquad \text{where } [S]=\begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0\\ -\nu_{12}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

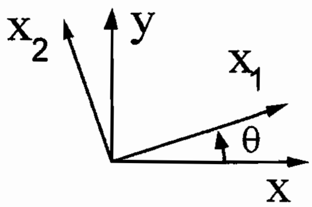
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}=\begin{bmatrix} Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix}$$

where
$$\begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21})\\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21})\\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{66} &= G_{12}\end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ, with respect to the (x, y) axes.



Then,
$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}=[T]\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix}=[T]^{-T}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{pmatrix}$$

where
$$[T]=\begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

and
$$[T]^{-T}=\begin{bmatrix}\cos^2\theta & \sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)\end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1,x_2) to (x,y)

$$[\overline{Q}]=[T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}]=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix} \quad \text{where}$$

$$\overline{Q}_{11}=Q_{11}C^4+Q_{22}S^4+2(Q_{12}+2Q_{66})S^2C^2$$
$$\overline{Q}_{12}=(Q_{11}+Q_{22}-4Q_{66})S^2C^2+Q_{12}\left(C^4+S^4\right)$$
$$\overline{Q}_{22}=Q_{11}S^4+Q_{22}C^4+2(Q_{12}+2Q_{66})S^2C^2$$
$$\overline{Q}_{16}=(Q_{11}-Q_{12}-2Q_{66})SC^3-(Q_{22}-Q_{12}-2Q_{66})S^3C$$
$$\overline{Q}_{26}=(Q_{11}-Q_{12}-2Q_{66})S^3C-(Q_{22}-Q_{12}-2Q_{66})SC^3$$
$$\overline{Q}_{66}=(Q_{11}+Q_{22}-2Q_{12}-2Q_{66})S^2C^2+Q_{66}\left(S^4+C^4\right)$$

with $C=\cos\theta$ and $S=\sin\theta$.

The compliance matrix $[S]\equiv[Q]^{-1}$ transforms to $[\overline{S}]\equiv[\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1,x_2) to (x,y), as

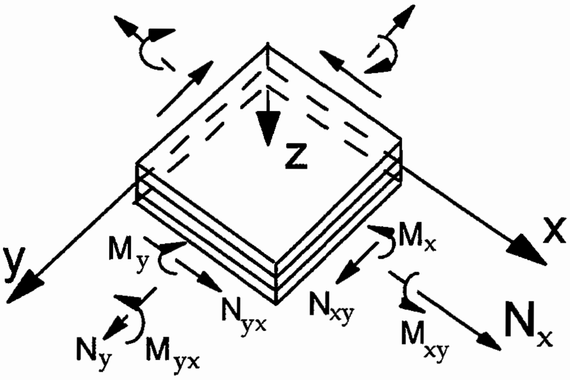
$$[\overline{S}]=[T]^T[S][T]$$

and in component form,

$$\overline{S}_{11}=S_{11}C^4+S_{22}S^4+(2S_{12}+S_{66})S^2C^2$$
$$\overline{S}_{12}=S_{12}\left(C^4+S^4\right)+(S_{11}+S_{22}-S_{66})S^2C^2$$
$$\overline{S}_{22}=S_{11}S^4+S_{22}C^4+(2S_{12}+S_{66})S^2C^2$$
$$\overline{S}_{16}=(2S_{11}-2S_{12}-S_{66})SC^3-(2S_{22}-2S_{12}-S_{66})S^3C$$
$$\overline{S}_{26}=(2S_{11}-2S_{12}-S_{66})S^3C-(2S_{22}-2S_{12}-S_{66})SC^3$$
$$\overline{S}_{66}=(4S_{11}+4S_{22}-8S_{12}-2S_{66})S^2C^2+S_{66}\left(C^4+S^4\right)$$

with $C=\cos\theta$, $S=\sin\theta$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $(\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0)^T$ and to a curvature $(\kappa_x, \kappa_y, \kappa_{xy})^T$. The stress resultants $(N_x, N_y, N_{xy})^T$ and bending moment per unit length $(M_x, M_y, M_{xy})^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^0 \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k dz = \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\overline{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts $i, j = 1, 2$ or 6.

Here,

$t =$ laminate thickness

$z_{k-1} =$ distance from middle surface to the inner surface of the k - th lamina

$z_k =$ distance from middle surface to the outer surface of the k - th lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $F_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

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October 2002

Engineering Tripos Part IIB: Module 4C2

Designing with Composites

Numerical answers 2005/6

1 (a) $A = \begin{bmatrix} 68.8 & 36 & 0 \\ 36 & 68.6 & 0 \\ 0 & 0 & 43.6 \end{bmatrix}$ MN/m

(b) Axial - 137.6 MN/m, Bending - 6900 Nm

2 (c) e.g. 5.55mm total, with mix of 65%:25%:10% of 0:90:45,

(d) Axial 500 MN/m, bending 1900Nm approx

3. 8 J/m approx

4. 0.76 MN/m