ENGINEERING TRIPOS PART IIB

Monday 1 May 2006 9 to 10.30

Module 4C7

RANDOM AND NON-LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

Candidates may bring their notebooks to the examination.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
CUED approved calculator allowed
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The heave (vertical) motion $x(t)$ of a floating oil production vessel is governed by the equation

$$\ddot{x} + 2\beta\omega_n\dot{x} + \omega_n^2 x = \alpha\eta(t)$$

where ω_n is the natural frequency, β is the damping ratio, α is a constant, and $\eta(t)$ is the random surface elevation of the ocean. The oil riser pipes attached to the vessel will be damaged if the heave motion exceeds the level $x = b$. The vessel is subjected to a storm of duration T in which the wave elevation $\eta(t)$ can be represented as white noise with double sided spectral density S_0.

(a) Show that the probability of damaging the oil riser pipes will be less than a specified value P providing the damping ratio satisfies the condition

$$\beta > -\left(\frac{\pi S_0 \alpha^2}{b^2 \omega_n^3}\right) \ln\left[-\left(\frac{2\pi}{\omega_n T}\right)\ln(1-P)\right]$$ [40%]

(b) Show that if P is small then the above inequality can be rewritten as

$$1 > -2\left(\frac{\sigma_x}{b}\right)^2 \ln\left(\frac{P}{N}\right)$$

where σ_x is the rms heave displacement and N is the number of heave cycles during the storm. [30%]

(c) For a particular storm N=2000, and $(b/\sigma_x) = 4$ when $\beta = 0.05$. Suggest a suitable value for the probability P and hence calculate the minimum damping ratio to avoid damage to the risers. How sensitive is the result for β to the value you have specified for P? [30%]

2 A random function $x(t)$ is defined by the equation

$$x(t) = A\sin(\omega t + \phi)$$

where ω is a specified frequency, and the amplitude A and phase ϕ are random variables (independent of time t). The phase has the probability density function

$$p(\phi) = \frac{1}{2\pi}, \qquad |\phi| \leq \pi$$

$$= 0, \qquad |\phi| > \pi$$

and the amplitude A is statistically independent of ϕ and has the mean squared value $\mathrm{E}[A^2] = \alpha$.

(a) Sketch an ensemble of realisations of the random function $x(t)$. [20%]

(b) Derive an expression for the autocorrelation function $R_{xx}(\tau)$. [20%]

(c) Define the terms "stationary" and "ergodic" and state whether these properties apply to the random function $x(t)$. [20%]

(d) A single degree of freedom system is excited by the random function $x(t)$ so that the equation of motion is

$$M\ddot{z} + B\dot{z} + Kz = x(t)$$

Solve for the steady state motion $z(t)$ for any particular realisation of the random function $x(t)$, and hence derive an expression for the autocorrelation function $R_{zz}(\tau)$. [30%]

(e) Is $z(t)$ ergodic? [10%]

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3 A nonlinear undamped vibratory system has the force-displacement characteristic shown in Fig. 1. The slope of the force-displacement characteristic is s for displacements with $|x| < b$.

(a) For a sinusoidal displacement of amplitude α (with $\alpha > b$) sketch the force waveform produced by the nonlinear element. [20%]

(b) Determine the Describing Function for the nonlinear element. Derive the limiting value of this function for very large amplitudes, and explain the physical significance of your result. [60%]

(c) If the system is driven by a force $f = a\cos(\omega t)$ derive an approximate equation that governs the response amplitude α. You are not required to solve this equation. [20%]

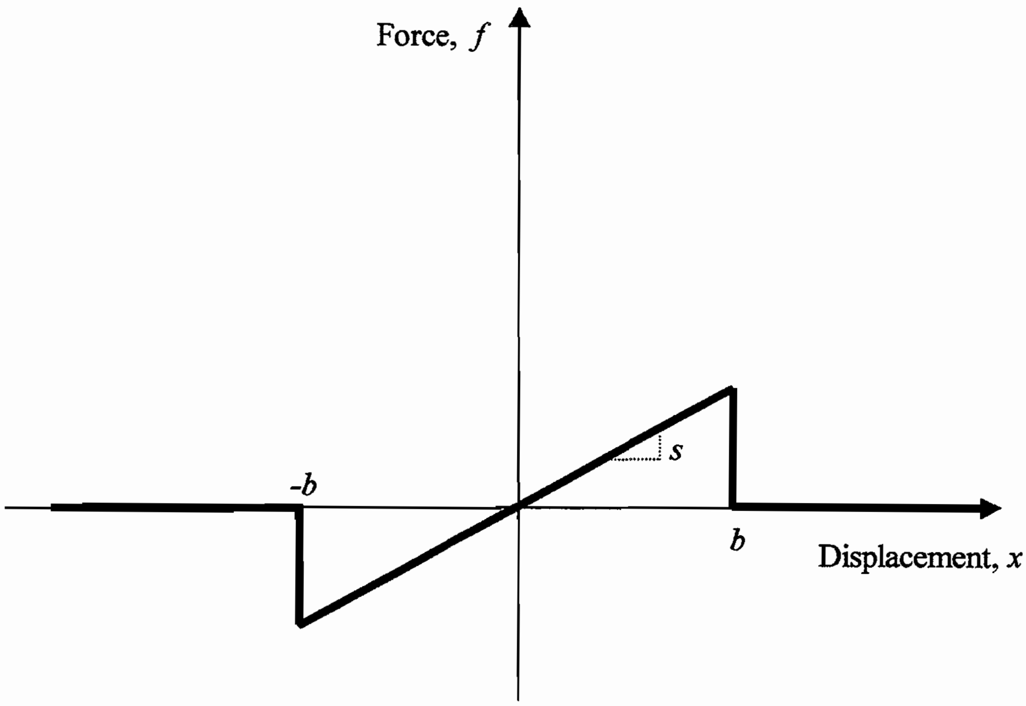

Fig. 1

4 A rigid pendulum of mass m and length l rotates with a uniform angular velocity Ω about a vertical axis, as shown in Fig. 2. The angle between the rotation axis and the pendulum is θ.

(a) Show that the equation of motion of the pendulum is given by

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \frac{\Omega^2}{2}\sin 2\theta = 0$$ [20%]

(b) Find the equilibrium or critical points for the system, and determine their type and stability. Comment specifically on how the stability of the equilibrium point $\theta = \dot{\theta} = 0$ depends on the rotation rate Ω. [40%]

(c) Sketch the behaviour of the system in the phase plane for $\Omega^2 > g/l$ and $-\pi \le \theta \le \pi$. [40%]

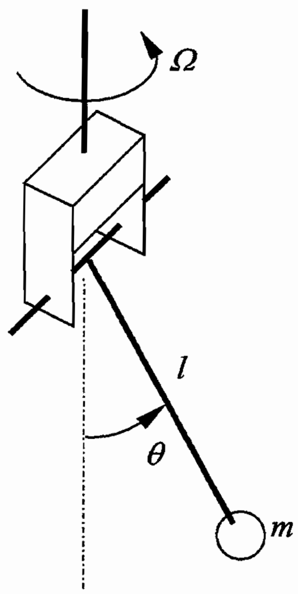

Fig. 2

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2006 Answers

1. (c) For example, if P=10^{-6} then we need β>0.134, and if P=10^{-7} then we need β>0.148.

2. (b) $R_{xx}(\tau) = (\alpha/2)\cos\omega\tau$; (c) stationary but not ergodic
   (d) $R_{zz}(\tau) = (\alpha/2)[(-\omega^2 M + K)^2 + (B\omega)^2]^{-1}\cos\omega\tau$
   (e) No.

3. (b) $D = (2s/\pi)\left[\pi/2 - \cos^{-1}(b/\alpha) - (b/\alpha)\sqrt{1 - b^2/\alpha^2}\right]$

4. Equilibrium/critical points:

$\theta = \dot{\theta} = 0$ stable for $\Omega^2 < g/l$
unstable for $\Omega^2 > g/l$

$\dot{\theta} = 0,\ \theta = \pm\pi$ always unstable

$\dot{\theta} = 0,\ \theta = \cos^{-1}(g/\Omega^2 l)$ stable, exists only for $\Omega^2 > g/l$