ENGINEERING TRIPOS PART IIB

Wednesday 26 April 2006 2.30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

Attachment: Special datasheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) (i) Show how the expression $\underline{a} \bullet (\underline{b} \times \underline{c})$ can be written in subscript notation.

The line of action of the force $F_i = (3,2,4)$ N is located by the vector $r_j = (2,1,3)$ m from the point A. Find the magnitude of the moment of the force about A in the direction of $a_k = (1,4,6)$. [25%]

(ii) If the vector v_k is related to the skew-symmetric tensor u_{ij} by the relation

$$v_k = e_{kji} u_{ij}$$

then express u_{ij} in terms of v_k. [25%]

(b) (i) In uniaxial tension, the *plastic* Poisson's ratio μ for an initially isotropic material is defined by

$$\varepsilon_y = \varepsilon_z = -\mu \varepsilon_x$$

where ε_x is the *total* strain produced by uniaxial tension σ_x, and $\varepsilon_y, \varepsilon_z$ are transverse strains. By assuming that the material is plastically incompressible, derive a formula for μ in terms of the secant modulus $E_{\mathrm{sec}}(\sigma_x)$, the Young's modulus E and the *elastic* Poisson's ratio ν of the material. [25%]

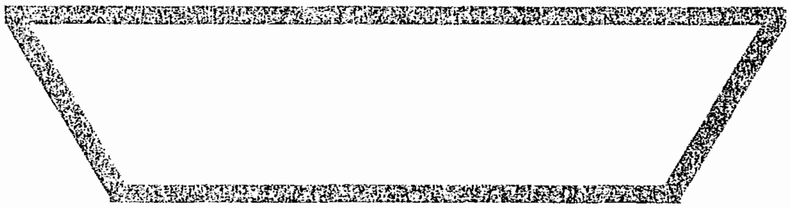
(ii) Using the result in (i) show that for multi-axial stress states, the total strain in J_2 deformation theory can be written as

$$\varepsilon_{ij} = \frac{1}{E_{\mathrm{sec}}(\sigma_{\mathrm{e}})}\left[(1+\mu)\sigma_{ij} - \mu \sigma_{kk} \delta_{ij}\right] \quad i,j = 1,2,3$$

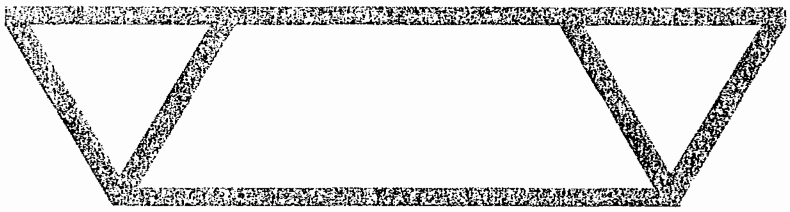
where now μ depends on the von Mises effective stress σ_{e} just as it depended on σ_x in the uniaxial test. [25%]

2 (a) A long torsion member is made in the form of a hollow box section with the cross-section illustrated in Fig. 1(a). The wall thickness t is uniform. The upper surface is of width $4a$ and the lower $3a$ and the height of the box section is $a\sqrt{3}/2$. Find an expression for the torsional stiffness of the member in terms of dimensions a and t and the shear modulus of the material.

The section is stiffened by adding two internal webs each of length a, as shown in Fig. 1(b). They are of the same material and also of thickness t. By what factor is the torsional stiffness thereby increased? [50%]



(a)



(b)

Fig. 1

(b) Show that a lower bound on the fully plastic torque T_P of any polygonal shaft may be calculated using the simple relation

$$T_P \geq \frac{2}{3} A_0 R k \,,$$

where A_0 is the cross-sectional area of the shaft, k is the yield stress in shear of the material and R is the radius of the largest inscribed circle that can be drawn within the cross-section of the shaft. [50%]

3 A metal-matrix composite component is manufactured by compacting metal particles around long, circular ceramic fibres of radius a. At a given instant, the body contains a small volume fraction f_{v} of fibres, and the uniaxial yield strength of the surrounding metal matrix is $\sigma_{\mathrm{y}}^{\mathrm{m}}$. In comparison with the metal matrix which yields according to the von Mises criterion, the ceramic fibres can be assumed to be rigid.

(a) When the composite is subjected to a pure hydrostatic state of stress Σ_{m} the radial velocity u^* within a representative unit cell of the composite, say of radius b, may be given by

$$u^* = Ar + B/r$$

where A and B are suitably chosen constants and r is the radial distance from the centre of the cell.

(i) Confirm that $b = a/\sqrt{f_{\mathrm{v}}}$. [10%]

(ii) Derive an upper bound, in terms of $\sigma_{\mathrm{y}}^{\mathrm{m}}$ and f_{v}, on the limit stress $\Sigma_{\mathrm{m}}^{\mathrm{L}}$ required to plastically deform the composite. [50%]

(iii) Comment of the values of $\Sigma_{\mathrm{m}}^{\mathrm{L}}$ when $f_{\mathrm{v}} \to 0$ and $f_{\mathrm{v}} \to 1$. [20%]

Note that $\int r\sqrt{1+a^4/r^4}\,\mathrm{d}r = \dfrac{a^2}{2}\int \sec^2\theta \csc\theta\,\mathrm{d}\theta$ when $\tan\theta = r^2/a^2$

(b) For an assumed velocity field within the representative unit cell of the composite with the form

$$u^* = (Cr + D/r)\sin 2\theta \quad \text{and} \quad v^* = (Cr + D/r)\cos 2\theta$$

in which C and D are suitably chosen constants, u^* is the radial velocity, v^* is the tangential velocity and θ is the angle that a radial line makes with the reference direction, show that the radial component of the compatible strain rate field is

$$\dot{\varepsilon}_r^* = C\left(1 + a^2/r^2\right)\sin 2\theta \ .$$

Obtain the corresponding expressions for $\dot{\varepsilon}_\theta^*$ and $\dot{\gamma}_{r\theta}^*$. [20%]

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$	$a_i e_i$
$\underline{a} \bullet \underline{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underline{c} = \underline{a} \times \underline{b}$	$c_i = e_{ijk} a_j b_k$
$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	e_{ijk} =1 when indices cyclic; = –1 when indices anticyclic and = 0 when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
trace$a$	tr$a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
gradϕ $= \nabla\phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underline{V}$	$V_{i,i}$
curl$\underline{V} \equiv \underline{\nabla} \times \underline{V}$	$e_{ijk} V_{k,j}$

Rotation of Orthogonal Axes

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - \mathrm{I}_1\sigma^2 + \mathrm{I}_2\sigma - \mathrm{I}_3 = 0$

$$\mathrm{I}_1 = \sigma_{ii} = \mathrm{tr}\sigma$$

$$\mathrm{I}_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

$$\mathrm{I}_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$$

$s^3 - \mathrm{I}_1' s^2 + \mathrm{I}_2' s - \mathrm{I}_3' = 0$

$$\mathrm{I}_1' = s_{ii} = \mathrm{tr}s \; ; \; \mathrm{I}_2' = \frac{1}{2}s_{ij}s_{ij} \; ; \; \mathrm{I}_3' = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

equilibrium $\quad \sigma_{ij},_i + b_j = 0$

small strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

compatibility $\quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$

equivalent to $\quad e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\dfrac{\partial^2\varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $\quad E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\quad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ij} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ij} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi,_2 = \frac{\partial\phi}{\partial x_2} \;,\; \sigma_{32} = -\phi,_1 = -\frac{\partial\phi}{\partial x_1} \text{ and } T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\dfrac{E}{2(1+\nu)}$	$\dfrac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\dfrac{E-2G}{2G}$	–	$\dfrac{(2G-E)G}{E-3G}$
E, λ	–	$\dfrac{E-\lambda+R}{4\lambda}$	$\dfrac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\dfrac{2G\nu}{1-2\nu}$
ν, λ	$\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\dfrac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\dfrac{G(3\lambda+2G)}{\lambda+G}$	$\dfrac{\lambda}{2(\lambda+G)}$		–

$R=\sqrt{E^2+2E\lambda+9\lambda^2}$

JAW

Answers

1 (a) $\frac{8}{\sqrt{53}}$ Nm $\quad u_{ij} = \frac{1}{2} e_{kji} v_k$

(b) $\mu = \frac{1}{2} - \frac{E_{\text{sec}}}{2E}(1-2\nu)$

2 (a) $4.08a^3 t$; $\quad 5.5\%$

3 $\frac{\Sigma_{\text{m}}^{\text{L}}}{\sigma_{\text{y}}^{\text{m}}} = \frac{1}{\sqrt{3}} \frac{f_{\text{v}}}{1-f_{\text{v}}} \left\{ \sqrt{1+(1/f_{\text{v}})^2} - \sqrt{2} + \ln\left(\frac{\tan(\theta_2/2)}{\tan(\pi/8)} \right) \right\}$ where $\tan\theta_2 = 1/f_{\text{v}}$

$\dot{\varepsilon}_\theta = -C\left(1 - a^2/r^2\right)\sin 2\theta$; $\quad \dot{\gamma}_{r\theta} = 2C\cos 2\theta$