ENGINEERING TRIPOS PART IIB

Saturday 6th May 2006 9 to 10.30

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the closed-loop system in Fig 1, where:

$$P(s) = \begin{bmatrix} G_1(s)(1+\Delta_1) & G_1(s)(1+\Delta_2) \\ G_2(s)(1+\Delta_1) & G_2(s)(1+\Delta_2) \end{bmatrix}, \qquad \|\Delta_i\|_\infty < \gamma_i$$

where $G_1(s), G_2(s)$ are SISO and stable. Denote the nominal system (when $\Delta_i = 0$, $i = 1,2$) by P_o. Assume K stabilises P_o.

(a) Determine a necessary and sufficient condition for the stability of the closed loop system for all Δ_i with $\|\Delta_i\|_\infty \le \gamma_i$ and $i = 1,2$. [40%]

(b) Let $\gamma = \max\{\gamma_1, \gamma_2\}$. Find a function $f(\cdot)$ such that

$$\|(I+PK)^{-1}\|_\infty < \|(I+P_oK)^{-1}\|_\infty \frac{1}{1-f(\gamma)}$$

where $f(\gamma) \to 0$ as $\gamma \to 0$. Note that f may also be a function of P_o and K. (Hint: you may find some of the Singular Value (in)equalities from the notes useful; a summary of those can be found on the next page). [50%]

(c) Discuss the implications of the inequality in part (b) regarding the energy of the input disturbance d and of the output y in Fig 1. [10%]

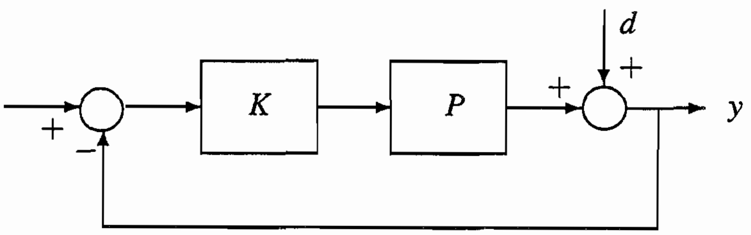

Fig. 1

The following results are taken from the lecture notes. For any matrices A, B, X, Y, Z of appropriate dimensions, and for which the indicated inverses exist:

- $\overline{\sigma}(A) - \overline{\sigma}(B) \leq \overline{\sigma}(A+B) \leq \overline{\sigma}(A) + \overline{\sigma}(B)$.

- $\overline{\sigma}\left(A^{-1}\right) = 1/\underline{\sigma}(A)$.

- $\underline{\sigma}(A) - \overline{\sigma}(B) \leq \underline{\sigma}(A+B) \leq \underline{\sigma}(A) + \overline{\sigma}(B)$.

- $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$.

- The non-zero eigenvalues of AB are equal to those of BA.

- $\sigma_i^2(XYZ) = \lambda_i(XYZZ^*Y^*X^*) = \lambda_i(YZZ^*Y^*X^*X)$.

(TURN OVER

2 (a) Consider an optimal control problem for the plant

$$\dot{x} = Ax + Bu,$$

where the objective is to minimize

$$J = \int_0^{t_f} (x^T Qx + u^T Ru)\, dt + x(t_f)^T X_{t_f} x(t_f)$$

for some $Q = Q^T$ and $R = R^T > 0$. Write down the Hamilton-Jacobi-Bellman (HJB) equation for this problem and find a solution in the form

$$V(x,t) = x^T X(t) x,$$

where $X(t)$ is the solution to a differential equation. [50%]

Hint: The HJB equation for the continuous time optimal control problem with incremental cost $c(x,u)$ is given by:

$$0 = \min_u \left\{ c(x,u) + \frac{dV}{dt} \right\}.$$

(b) A unit mass moves on a straight line from a given non-zero initial position and velocity. It is to be brought to rest at the origin at time $t = 1$. Explain in detail how you would use your answer to part (a) to find the force $u(t)$ that achieves this whilst minimizing

$$\int_0^1 u(t)^2\, dt$$

(You should set up the problem carefully, not leaving it in matrix form, but you do not need to solve any differential equations you write down.) [50%]

3 Consider a linear system with transfer function $G(s)$.

(a) Define the H_2 and H_∞ norms of this system. When do they exist? [20%]

(b) If $\hat{y}(s) = G(s)\hat{u}(s)$, with $u \in L_2$, how is the relationship between u and y constrained by $\|G(s)\|_2$ and $\|G(s)\|_\infty$? [20%]

(c) Suppose $G(s)$ has a controllable and observable state-space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

By considering the quadratic form $\frac{d}{dt}(x^T X x) + y^T y$ where X is constant (or otherwise) derive a formula for $\|G(s)\|_2$ in terms of the matrices A, B and C and the solution to a matrix equation. [60%]

END OF PAPER