ENGINEERING TRIPOS PART IIB

Monday 8 May 2006 2.30 to 4

Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A predictive controller with constraints has been designed for the following discrete-time system:

$$x(k+1) = Ax(k) + Bu(k)$$

Let x_s and u_s be the prediction of the state and input, respectively, at time $k+s$ when the state at time k is $x_0 = x(k)$. The vectors U and X are defined as

$$U := \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad X := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix},$$

where $x_{s+1} = Ax_s + Bu_s$ over the horizon $s = 0, 1, \ldots, N-1$.

The prediction matrices Φ and Γ such that $X = \Phi x_0 + \Gamma U$ are given by

$$\Phi = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 5 & 8 \\ 8 & 13 \\ 21 & 34 \\ 34 & 55 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 0 \\ 13 & 3 & 1 \\ 21 & 5 & 1 \end{pmatrix}.$$

(a) Show that the length of the control horizon is $N = 3$. [10%]

(b) What are the values of A^3, A^2B and AB? [30%]

(c) Show that the constraint:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} U \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix} x(k)$$

is equivalent to the constraint: [10%]

$$x_1 \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(cont.

(d) Compute the values for a, b, c, d, e, f, g and h such that the constraints:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ c & 1 & 0 \\ d & -1 & 0 \\ 0 & c & 1 \\ 0 & d & -1 \\ e & f & 1 \end{pmatrix} U \leq \begin{pmatrix} a \\ b \\ a \\ b \\ a \\ b \\ a \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(k-1) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ g & h \end{pmatrix} x(k)$$

are equivalent to the following constraints on the input rate and terminal state:

$$-1 \leq \Delta u_s \leq 2, \quad s = 0, 1, \ldots, N-1$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} x_N \leq 2,$$

where $\Delta u_s = u_s - u_{s-1}$, for $s = 0, 1, \ldots, N-1$ and $u_{-1} = u(k-1)$. [50%]

2 A predictive controller is to be designed for the following discrete-time system:

$$x(k+1) = Ax(k) + B[u(k) + d(k)],$$
$$y(k) = Cx(k)$$

where $x(k)$ is the state, $u(k)$ is the input, $y(k)$ is the measured output, and $d(k)$ is an unmeasured disturbance acting on the input.

(a) Rewrite the equations using the augmented state vector $[x(k)^T, d(k)^T]^T$, assuming that the disturbance $d(k)$ is constant. [5%]

(b) Recall that (C, A) is detectable if and only if

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}$$

has full column rank for all $\lambda \in \Lambda$, where Λ is the set of eigenvalues of A on or outside the unit circle. If A is stable, show that the augmented system is detectable if and only if the following matrix has full column rank: [25%]

$$\begin{pmatrix} I - A & -B \\ C & 0 \end{pmatrix}.$$

(c) The controller is to drive the output to a given constant reference r. The system is subject to input disturbances and input constraints. An offset-free target equilibrium pair is found by solving the following constrained optimization problem:

$$J(\hat{d}, r) = \min_{u,x} (y - r)^T (y - r)$$

subject to the constraints:

$$x = Ax + B(u + \hat{d}),$$
$$u_{\text{low}} \le u + \hat{d} \le u_{\text{high}},$$
$$u_{\text{low}} \le u \le u_{\text{high}},$$

where $\hat{d}$ is the current estimate of the input disturbance.

(cont.

(i) Suppose $J(\hat{d}, r) = 0$. Show that the solution to the optimization problem is unique if and only if the following matrix has full column rank: [30%]

$$\begin{pmatrix} I - A & -B \\ C & 0 \end{pmatrix}.$$

(ii) Suppose $A = 0.5$, $B = 1$ and $C = 1$, $u_{\text{low}} = -3$ and $u_{\text{high}} = 3$. If $\hat{d} = 2$, show that $J(\hat{d}, r) > 0$ if $r < -2$ or $r > 6$. [40%]

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3 A *relay with dead-band*, shown in Fig. 1, has input e and output u, with the input-output relationship:

$$u = \begin{cases} +1 & \text{if} \quad e > d \\ 0 & \text{if} \quad |e| \leq d \\ -1 & \text{if} \quad e < -d \end{cases}$$

(a) Show that if $e(t) = E\sin(\omega t)$ and $E > d$ then its describing function is given by [30%]

$$N(E) = \frac{4}{\pi E}\sqrt{1 - \left(\frac{d}{E}\right)^2}$$

(b) What is its describing function if $E < d$? [5%]

(c) Verify that $N(E)$ has a stationary point at $E = d\sqrt{2}$, and hence sketch the graph of $N(E)$. [30%]

(d) The relay with dead-band is in a negative feedback loop around a linear system as shown in Fig. 2. The linear system has transfer function

$$G(s) = \frac{k}{s(s+1)^2} \qquad (k > 0)$$

Show that the describing function method predicts the absence of limit-cycle oscillations if $k < \pi d$. [35%]

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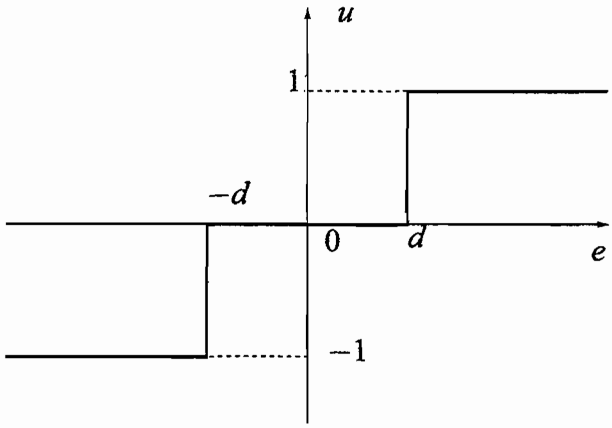

Fig. 1

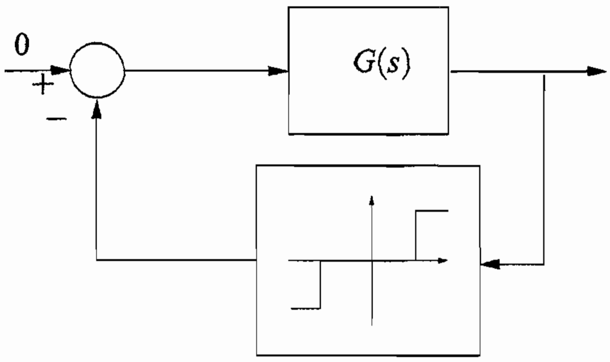

Fig. 2

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4 (a) When studying systems of the form $\dot{x} = f(x)$ it is common to assume that the function $f(.)$ is *Lipschitz continuous*. Explain what this means, and why such an assumption is made. [30%]

(b) Explain how *LaSalle's theorem* extends Lyapunov's direct method for establishing asymptotic stability. [30%]

(c) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2) \end{aligned}$$

where $h_1(0) = h_2(0) = 0$ and $yh_1(y) > 0$ and $yh_2(y) > 0$ for $0 < |y| < Y$, where Y is given. Both $h_1(.)$ and $h_2(.)$ are Lipschitz continuous. By considering the function

$$V(x) = \int_0^{x_1} h_1(y)dy + \frac{1}{2}x_2^2$$

show that the origin is an asymptotically stable equilibrium state of the system. [40%]

END OF PAPER