ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Friday 5 May 2006 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed
Candidates may bring their notebooks to the examination

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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1 (a) (i) Prove the identity

$$(\mathbf{e}\times\mathbf{x})\cdot(\mathbf{e}\times\mathbf{x})=x_q x_q - x_p x_q e_p e_q$$

where **e** is a fixed unit vector, and **x** is the position vector. [15%]

(ii) Hence use suffix notation to show that

$$\nabla\cdot\left[\frac{\mathbf{e}\times\mathbf{x}}{|\mathbf{e}\times\mathbf{x}|}\right]=0.$$ [30%]

What is the geometrical significance of the vector $\dfrac{\mathbf{e}\times\mathbf{x}}{|\mathbf{e}\times\mathbf{x}|}$? [5%]

(b) In a volume V of empty space, bounded by a surface S, the electrostatic potential $\phi(\mathbf{x})$ is determined by a condition of minimum energy. This requires that the integral

$$I=\iiint_V \nabla\phi\cdot\nabla\phi\, dV$$

is minimised. Use the method of the calculus of variations to show that $\phi(\mathbf{x})$ satisfies Laplace's equation everywhere in the volume V. If there is no constraint on the value of $\phi(\mathbf{x})$ on the boundary surface S, what boundary condition must be satisfied there? [50%]

2 (a) The position of a point on the surface of a sphere of radius a can be determined by the two polar angles θ, ϕ as defined in Fig. 1. A path on the surface of the sphere can then be described by a function $\theta = f(\phi)$. Show that the path of shortest length between two given end-points is determined by minimising the integral

$$\int_{\phi_1}^{\phi_2} \sqrt{f'^2 + \sin^2 f}\; d\phi$$

where ϕ_1, ϕ_2 are the values of ϕ at the end points. [30%]

(b) Find a differential equation which must be satisfied by the function f. Under what conditions is $f =$ constant a solution to this equation? [40%]

(c) Hence explain why a path following the "great circle" through the two end points always satisfies the minimum-length condition. (A great circle is a curve on the sphere which lies in a plane which passes through the centre of the sphere.) [30%]

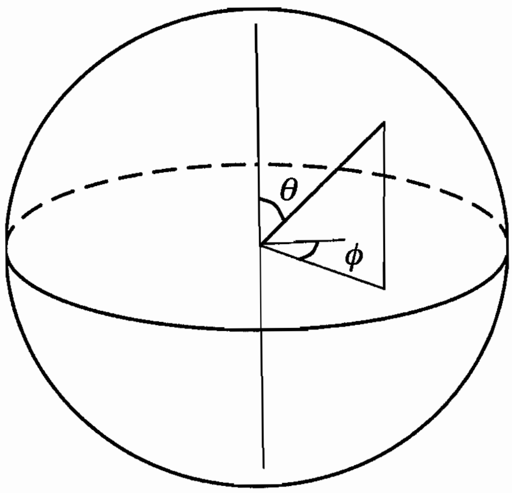

Fig. 1

3 For an equation $u_{xx} - 3\,u_{xy} + 2\,u_{yy} = 0$:

(a) determine the type of the equation; [10%]

(b) solve its characteristic equation, and if any real characteristics exist sketch them in the x-y plane; [20%]

(c) find its canonical form and the most general solution; [20%]

(d) solve the PDE with boundary conditions:

$$u(x,0) = 1,\ \ u_y(x,0) = 1 \text{ at } y = 0 \text{ for } x \in (-\infty, +\infty);$$ [30%]

(e) if $u(x,0) = 1$ and $u_y(x,0) = 1$ are only defined at $y = 0$ for $x \in [-1,1]$, find $u(-1,2)$ and $u(1,2)$. [20%]

4 A one-dimensional unsteady heat conduction problem

$$T_t - k\,T_{xx} = f(x), \qquad x \in [0, \infty),\ \ t > 0$$

is subject to initial condition $T(x,0) = 0$.

(a) Assume that the end $x = 0$ is perfectly insulated. Specify the correct boundary condition at $x = 0$. [10%]

(b) What is the corresponding partial differential equation for the Green's function of this problem? Describe possible method(s) for solving this problem. Use the information given in the notes to find the Green's function as the fundamental solution.

[30%]

(c) Show that the problem has a solution which can be expressed as

$$T(x,t) = \int_0^\infty\!\!\int_0^\infty G(x,t,x_0,t_0)\,f(x_0)\,dx_0\,dt_0\,,$$

where $G(x, t, x_0, t_0)$ is the Green's Function of this problem. [30%]

(d) Show that this solution is unique. [30%]

END OF PAPER