ENGINEERING TRIPOS PART IIB
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Thursday 27 April 2006 9 to 10.30

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

Answer not more than ***three*** *questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to Sections A and B should be tied together and handed in separately.

Attachment:

4M13 datasheet (4 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (a) Distinguish between the notions of a branch point and a simple pole for a function of a complex variable. Illustrate your answer with an example of each. [25%]

(b) Explain, with an example, the idea of the radius of convergence of a complex function. [15%]

(c) Evaluate by contour integration and the residue theorem the integral

$$\int_0^{\infty} \frac{dx}{x^4 + a^4}$$

where a is a real constant. [60%]

2 (a) Under what circumstances can a Taylor series expansion and a Laurent series expansion be used to express a function of a complex variable? How can such expansions be used to perform the process of *analytic continuation* of a complex function? [30%]

(b) Explain the use of Jordan's lemma in contour integration. [20%]

(c) Find and classify all singularities of the following functions:

(i) $f(z) = \dfrac{(z-4)\,z^{1/3}}{(z^2 - 6z + 8)}$ [15%]

(ii) $f(z) = \dfrac{z \ln(z)}{\sin(z)}$ [20%]

(iii) $f(z) = (z^2 - z)^{1/2}$ [15%]

In each case, calculate the residues at all poles.

SECTION B

3 The operating cost f of an electricity supply system varies with the voltage V (in kV) and the conductance C (in S) as

$$f = \frac{2\times10^5}{V^2 C} + 4\times10^3 C + V$$

(a) Use the standard optimality conditions for an unconstrained optimization problem to find the voltage and conductance that minimize the operating cost. [25%]

(b) Apply the Steepest Descent Method to this problem for two iterations starting from an initial solution $(V,C) = (100, 0.1)$. Evaluate the step length α_k using the formula given in the 4M13 datasheet rather than by line search. [40%]

(c) In the light of your answer to (a) comment on and account for the performance of the Steepest Descent Method observed in (b). [20%]

(d) Comment on the relative advantages and disadvantages of using Newton's Method and the Conjugate Gradient Method rather than the Steepest Descent Method on unconstrained minimization problems. [15%]

(TURN OVER

4 A rectangular platinum gauze catalyst must provide at least 240 cm^2 of effective area. The mounting covers the periphery of the gauze, as shown schematically in Fig. 1, reducing the effective area from xy cm^2 to $(x-6)(y-4)$ cm^2.

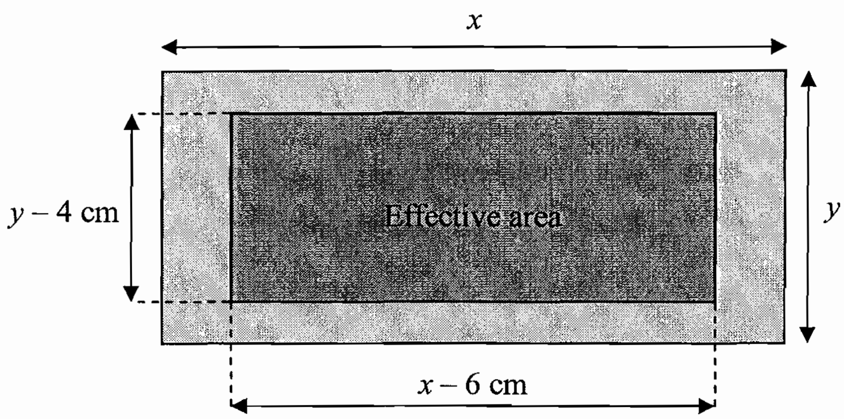

Figure 1

It is desired to minimize the cost of the catalyst, which is proportional to the total area of platinum gauze used.

(a) (i) Formulate this design task as a constrained minimization problem with two control variables, x and y. [5%]

(ii) Assuming the constraint is active at the optimum, eliminate x from your expression for the objective function, thus creating an unconstrained, univariate minimization problem. [10%]

(iii) Estimate, using a Golden Section line search, the value of y that minimizes the cost of the catalyst. A suitable initial interval for y is between 15 and 21 cm. The search can be halted when the interval has been reduced four times. [35%]

(b) By using a suitable Kuhn-Tucker multiplier formulation of the original constrained optimization problem in part (i) of (a), identify the three equations that give the first-order conditions that must be satisfied at an optimum, and hence show that the constraint must be active at the optimum. [20%]

(c) Solve these three equations and verify that the solution represents a minimum. Hence comment on the performance of the Golden Section line search in part (iii) of (a). [30%]

END OF PAPER

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OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

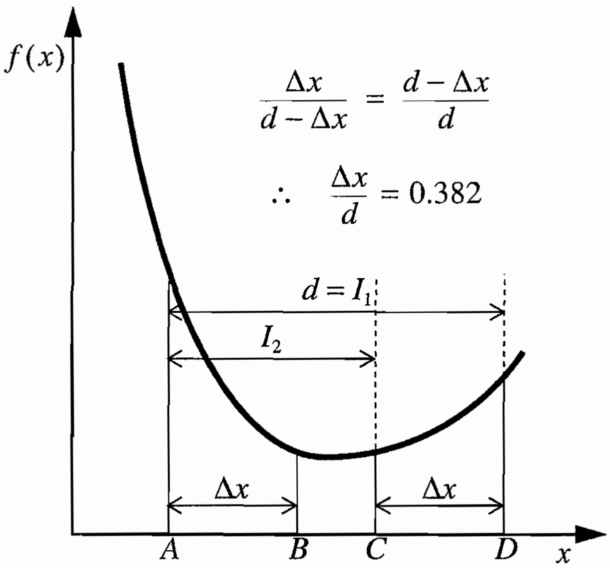
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\boldsymbol{H}(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T\boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T\boldsymbol{H}(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T\boldsymbol{H}(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,\boldsymbol{J}(\mathbf{x}_k)^T\boldsymbol{J}(\mathbf{x}_k)\,]^{-1}\,\boldsymbol{J}(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } \boldsymbol{J}(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \left[\nabla h_1(\mathbf{x}^*) \ \ldots \ \nabla h_m(\mathbf{x}^*) \right] = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \le 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \ge 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \leq 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$

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Numerical Results

1 (a) -
(b) -
(c) $\dfrac{\pi}{2\sqrt{2}\,a^3}$

2 (a) -
(b) -
(c) (i) Pole at $z = 2$
Branch cut at $z = 0$
Residue is $\sqrt[3]{2}$
(ii) Poles at $z = \pm n\pi$ for all integer $n \neq 0$
Branch point at $z = 0$
Residue is $\dfrac{n\pi \ln(n\pi)}{(-1)^n}$
(iii) Branch points at $z = 0, 1$

3 (a) 237.8 kV, 0.02973 S
(b) First two iterations go to $(V, C) = (100, 0.05)$ and $(100, 0.0625)$
(c) -
(d) -

4 (a) (i) -
(ii) -
(iii) $16.42 \text{ cm} \leq y \leq 17.29 \text{ cm}$
(b) -
(c) $y = 16.649$ cm, $x = 24.874$ cm, $\mu = 1.316$