2007 IIB 4A12 Prof R E Britter
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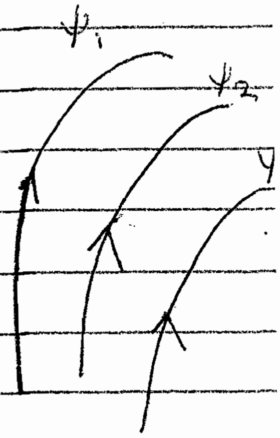
4A12

(1) (a) If α is small,



$$\underline{u}\cdot\nabla T \approx 0$$

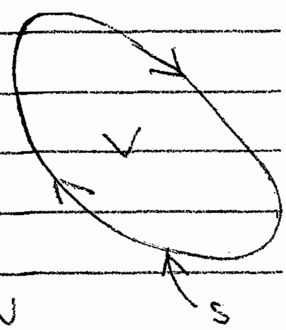
$\Rightarrow$ T conserved along streamlines

$$\Rightarrow \quad \underline{\underline{T = T(\psi)}}$$

$$\Rightarrow \quad \nabla T = T'(\psi)\,\nabla\psi$$

$$\Rightarrow \quad \underline{u}\cdot\nabla T = \alpha\nabla\cdot(\nabla T) = \alpha\nabla\cdot(T'(\psi)\nabla\psi)$$

Integrating of volume V



$$\int_V \underline{u}\cdot\nabla T\,dV = \alpha\int_V \nabla\cdot(T'(\psi)\nabla\psi)\,dV$$

$$\Rightarrow \quad \int_V \nabla\cdot(T\underline{u})\,dV = \alpha\int_V \nabla\cdot(T'(\psi)\nabla\psi)\,dV$$

Use Gauss to convert to surface integrals,

$$\oint_S T\underline{u}\cdot d\underline{S} = \alpha\oint_S T'(\psi)\nabla\psi\cdot d\underline{S}$$

But $\underline{u}\cdot d\underline{S} = 0$ and $T'(\psi)$ is constant on S.

Thus

$$\underline{\underline{\alpha T'(\psi)\oint \nabla\psi\cdot d\underline{S} = 0}}$$

(b) Using Gauss again we have $\alpha T'(\psi)\int \nabla^2\psi\,dV = 0$

But $\nabla^2\psi = -\omega$

c.f. $\underset{\sim}{u} = \left[\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right]$

$$\underset{\sim}{\omega} = \nabla\times\underset{\sim}{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\underset{\sim}{e}}_z = \left(-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\hat{\underset{\sim}{e}}_z$$

$\Rightarrow \quad \alpha T'(\psi)\int \omega \, dV = 0$

$\Rightarrow \quad \alpha T'(\psi)\oint \underset{\sim}{u}\cdot d\underset{\sim}{l} = 0$ by Stokes.

But $\alpha \neq 0$ and $\oint \underset{\sim}{u}\cdot d\underset{\sim}{l} \neq 0$

Thus $\underline{\underline{T'(\psi) = 0}} \Rightarrow \underline{\underline{T = \text{constant}}}$

This is achieved by a slow, cross-stream diffusion of heat which continues until T is perfectly uniform.

(c) The P-B theorem states that, if the flow is steady, 2D laminar, with closed streamlines and high Re, then ω is uniform. Reason:

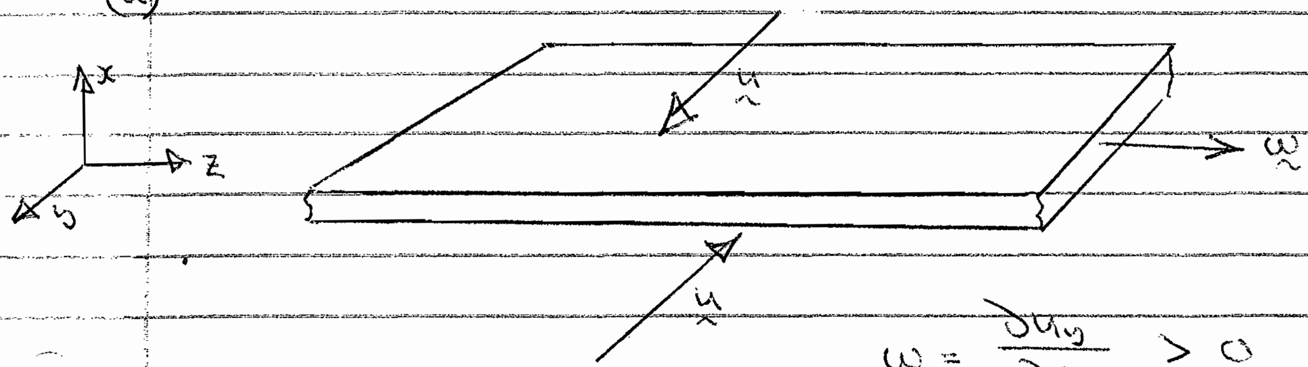
- 2D Vorticity equation, $\frac{D\omega}{Dt} = \nu\nabla^2\underset{\sim}{\omega}$, is same as heat equation if $\alpha = \nu$. Thus ω behaves like T.
- ν small implies $\underset{\sim}{u}\cdot\nabla\omega \simeq 0 \Rightarrow \omega = \omega(\psi)$

  c.f. $T = T(\psi)$
- ν finite means a slow but finite diffusion eventually eradicates any cross-stream gradients in ω.

(2) $\underset{\sim}{\omega} = (u_0/\delta)\exp[-x^2/\delta^2]\,\hat{\underset{\sim}{e}}_z \quad , \quad \delta = \delta(t)$

$\underset{\sim}{u} = u_y(x,t)\,\hat{\underset{\sim}{e}}_y$

(a)



$\omega_z = \dfrac{\partial u_y}{\partial x} > 0$

(b) $\dfrac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega}$

$\Rightarrow \dfrac{\partial \underset{\sim}{\omega}}{\partial t} + \cancel{u_y \dfrac{\partial}{\partial y}\underset{\sim}{\omega}} = \cancel{\omega_z \dfrac{\partial}{\partial z}\underset{\sim}{u}} + \nu \dfrac{\partial^2 \underset{\sim}{\omega}}{\partial x^2}$

$$\Rightarrow \underline{\underline{\dfrac{\partial \underset{\sim}{\omega}}{\partial t} = \nu \dfrac{\partial^2 \underset{\sim}{\omega}}{\partial x^2}}}$$

$\Rightarrow u_0 \dfrac{\partial}{\partial \delta}\left[\dfrac{1}{\delta}\exp(-x^2/\delta^2)\right]\delta'(t) = \nu \dfrac{\partial^2}{\partial x^2}\left[\dfrac{u_0}{\delta} e^{-x^2/\delta^2}\right]$

$\Rightarrow u_0\delta'(t)\left[-\dfrac{1}{\delta^2} + \dfrac{2x^2}{\delta^4}\right]e^{-x^2/\delta^2} = \nu\dfrac{u_0}{\delta}\dfrac{\partial}{\partial x}\left[-\dfrac{2x}{\delta^2}e^{-x^2/\delta^2}\right]$

$\Rightarrow u_0\delta'(t)\,e^{-x^2/\delta^2}\left[-\dfrac{1}{\delta^2} + \dfrac{2x^2}{\delta^4}\right] = 2\nu\dfrac{u_0}{\delta}\left[-\dfrac{1}{\delta^2} + \dfrac{2x^2}{\delta^4}\right]e^{-x^2/\delta^2}$

$\Rightarrow \delta'(t) = 2\nu/\delta$

$$\Rightarrow \underline{\underline{\delta\delta'(t) = 2\nu}}$$

(c)

$$\frac{d\delta^2}{dt} = 4\nu$$

$$\Rightarrow \quad \delta^2 = \delta_0^2 + 4\nu t$$

$$\Rightarrow \quad \delta = \sqrt{\delta_0^2 + 4\nu t}$$

The sheet is diffusing, with diffusion length $\sim \sqrt{4\nu t}$

The net flux of vorticity per unit length of the sheet is,

$$\Phi = \int_{-\infty}^{\infty} \omega_z \, dx = \frac{u_0}{\delta} \int_{-\infty}^{\infty} \exp[-x^2/\delta^2] \, dx$$

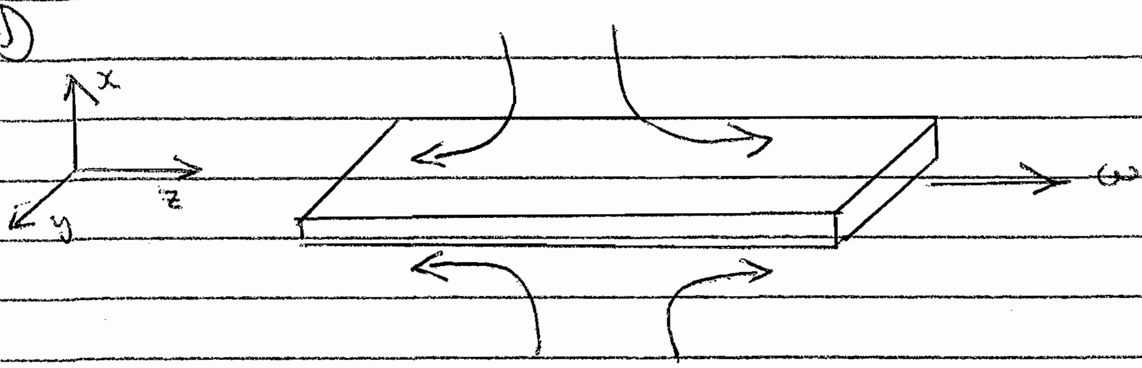
$$= u_0 \int_{-\infty}^{\infty} e^{-p^2} dp \qquad p = x/\delta$$

Thus $\Phi \propto u_0$. This flux cannot be changed by diffusion

c.f. $$\frac{\partial \omega_z}{\partial t} = \nu \frac{\partial^2}{\partial x^2} \omega_z$$

$$\Rightarrow \quad \frac{d\Phi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \omega_z \, dx = \nu \left[\frac{\partial \omega_z}{\partial x} \right]_{-\infty}^{\infty} = 0$$

(d)



The sheet is now subject to stretching by the irrotational straining flow.

Physical Processes:

1. Diffusion causes δ to increase and $\omega_{max} \sim \frac{u_0}{\delta}$ to fall
2. Advection by the straining flow causes δ to decrease.
3. Stretching by the straining flow causes ω_{max} to increase

With the correct choice of α, we can get a steady flow

If $\begin{cases} \alpha \gg \nu/\delta_0^2 & \text{the sheet will thin and } \omega_{max} \text{ rise,} \\ \alpha \ll \nu/\delta_0^2 & \text{the sheet will thicken and } \omega_{max} \text{ fall} \end{cases}$

QUESTION 3

(a) BOOKWORK – TO COME

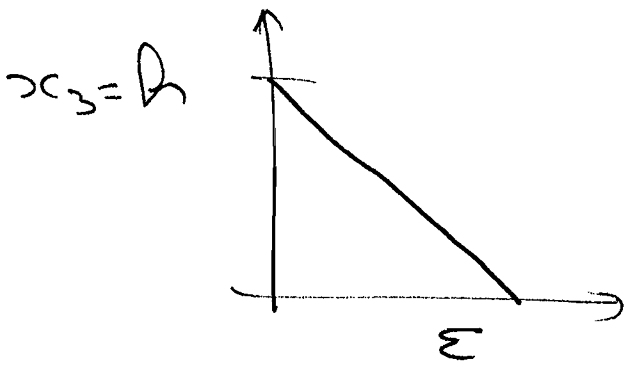
(b)(i) NO MEAN WIND

$$\therefore \quad -\nu \frac{\partial u_i}{\partial x_k} \cdots\cdots + \frac{g}{T}\overline{\theta u} = 0$$

$$\downarrow$$

$$-\varepsilon + \frac{g}{T}\frac{Q}{\rho c_p} = 0$$

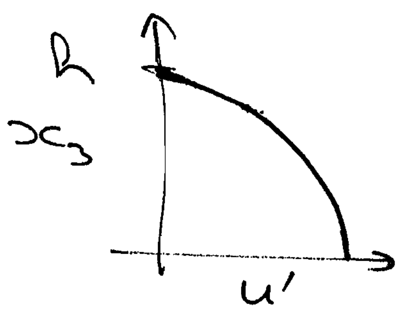
$$\therefore \quad \varepsilon = Q_0\left(1 - x_3/h\right)$$



(ii) $l \sim h$

$$\varepsilon \sim u'^3/l \sim Q_0\left(1 - x_3/h\right) \sim u'^3/h$$

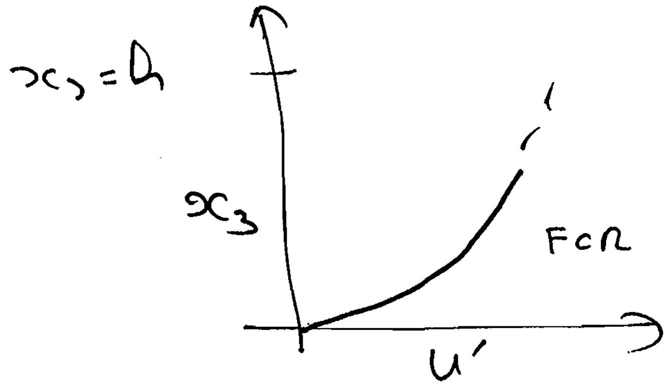
$$\therefore u' = \left\{Q_0 h\left(1 - x_3/h\right)\right\}^{1/3}$$



(iii) $l \sim x_3$

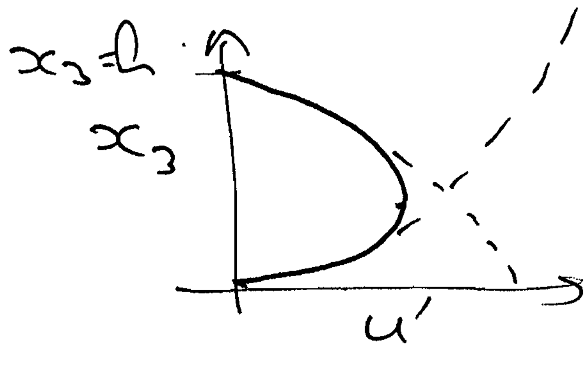
$$\varepsilon \sim u'^3/l \sim Q_0(1 - x_3/h) \sim u'^3/x_3$$

$$\therefore u' \sim \left(Q_0 x_3 (1 - x_3/h)\right)^{1/3}$$



FOR x_3/h SMALL

COMPOSITE



QUESTION 4

(i) INCREASE IN ENERGY $= \frac{1}{2} m_1 V^2 = m_1 g h$

$= 20 \times 10^{-6} \times 1000 \times 9.81 \times 0.2$

$= 3.92 \times 10^{-2}$ JOULE

(ii) TOTAL ENERGY

$= 3.92 \times 10^{-2}$ JOULE

NOW IN $250\ cm^3$ ~~of~~ OF WATER

$\therefore \frac{1}{2} m_2 . 3u'^2 = 3.92 \times 10^{-2}$

$\frac{1}{2} \times 250 \times 10^{-6} \times 1000 \times 3 \times u'^2 = 3.92 \times 10^{-2}$

$\therefore u'^2 = 0.105\ m^2/s^2$, $u' = 0.323\ m/s$

$Re = \frac{u' l}{\nu} = \frac{0.323 \times 0.07}{1 \times 10^{-6}} = 2.26 \times 10^4$

(iii) $\varepsilon \sim u'^3 / l = (0.323)^3 / 0.07 = 0.481 \left(m^2/s^2\right) /$

NOTE IF A LINEAR DECAY AT THIS RATE THE $u'^2 = 0$ WHEN $t = \frac{0.105}{0.481} = 0.2$ SECOND

(iv) $\frac{d}{dt} \frac{3}{2} u'^2 = - u'^3/l$

l = constant = 0.07 m

$k = \frac{3}{2} u'^2$

$\therefore \frac{dk}{dt} = -\left(\frac{2}{3} k\right)^{3/2} / l$

$k^{-3/2}\, dk = -\left(2/3\right)^{3/2} \frac{dt}{l}$

$-2 k^{-1/2} = -\left(2/3\right)^{3/2} t/l$ + const.

$\therefore k^{-1/2} = \left(2/3\right)^{3/2} t/l$ + const.

at $t = 0$ $k = k_0$

$\therefore k^{-1/2} = \left(2/3\right)^{3/2} t/l + k_0^{-1/2}$ | LEAVE AS SOLUTION IN THIS FORM.

At Re. = 100 ; $\frac{u' l}{\nu} = 100$.

$u' = \frac{100 \times 1 \times 10^{-6}}{0.07} = 1.43 \times 10^{-3}$ m/s.

$u'^2 = 2.04 \times 10^{-6}$ m^2/s^2

$\therefore k = \frac{3}{2} u'^2 = 3.06 \times 10^{-6}$ m^2/s^2

$\{3.06 \times 10^{-6}\}^{-1/2} = \left(2/3\right)^{3/2} t/l + \left(0.1575\right)^{-1/2}$

$0.57 \times 10^3 = \frac{0.544}{0.07} t + 2.52$

$t = \frac{567.5 \times 0.07}{0.544} = 73.0$ seconds.

(v) Initially $\eta = \left(\nu^3/\varepsilon\right)^{1/4}$

$$= \left[\frac{(1\times10^{-6})^3}{0.481}\right]^{1/4} = 3.8\times10^{-5}\ \text{m}.$$

Finally $\eta = \left(\nu^3/\varepsilon\right)^{1/4}$

$$\varepsilon = u'^3/\ell = (0.00143)^3/0.07$$

$$\eta = \left(\frac{(1\times10^{-6})^3\times 0.07}{(0.00143)^3}\right)^{1/4}$$

$$= 2.21\times10^{-3}$$

(vi) Temperature rise due to dissipation $= \dfrac{3.92\times10^{-2}\ \text{Joule}}{m\, C_p}$

$$= \frac{3.92\times10^{-2}\ \text{Joule}}{250\times10^{-6}\times1000\times4{,}200\ \text{J/kg K}}$$

$$= 3.7\times10^{-6}\ \text{Kelvin}$$