2007 IIB 4C7 Prof R Langley
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4C7 2007 CRIB

1 a) $\ddot{x} + 2\beta\omega_n\dot{x} + \omega_n^2 x = F(t)$; $\omega_n = 250\times2\times\pi$ rad/s $\beta = 0.02$

Standard equations $\sigma_x^2 = \dfrac{\pi S_0}{2\beta\omega_n^3} = \dfrac{\pi\times2}{2\times0.02\times(250\times2\times\pi)^3} = \underline{4.05\times10^{-8}}\ m^2$

$\sigma_{\dot{x}}^2 = \dfrac{\pi S_0}{2\beta\omega_n} = \underline{0.1}\ m^2/s^2$

For stress, $S = 2.5\times10^4 x \Rightarrow \sigma_S = 2.5\times10^4\sigma_x = 2.5\times10^4(4.05\times10^{-8})^{1/2} = \underline{5.04}$ MPa

$\sigma_{\dot{S}} = 2.5\times10^4\sigma_{\dot{x}} = \underline{7.9\times10^3}$ MPa s^{-1}

For $b = 25$, $\nu_b^+ = \dfrac{1}{2\pi}\dfrac{\sigma_{\dot{S}}}{\sigma_S} e^{-\frac{1}{2}(b/\sigma_S)^2} = 0.0011$

$P = 1 - e^{-\nu_b^+ t} = 1 - e^{-0.0011\times5\times60} = \underline{0.28}$ [35%]

t: 5 mins = 5×60 secs.

b) Damage formula: $D = \nu_0^+ T\int_0^\infty \dfrac{p(S)}{N(S)}dS$, where $p(S) = \dfrac{S}{\sigma^2}e^{-\frac{1}{2}(S/\sigma)^2}$ $\sigma \equiv \sigma_S$

Now $N(S) = 8\times10^7 S^{-3} \Rightarrow D = \dfrac{\nu_0^+T}{8\times10^7}\underbrace{\int_0^\infty \dfrac{S^4}{\sigma^2}e^{-\frac{1}{2}(S/\sigma)^2}dS}_{\frac{1}{2}\sqrt{2\pi}\,\sigma\times3\sigma^4} = \dfrac{\nu_0^+T}{8\times10^7}\sqrt{\dfrac{\pi}{2}}\,3\sigma^3$

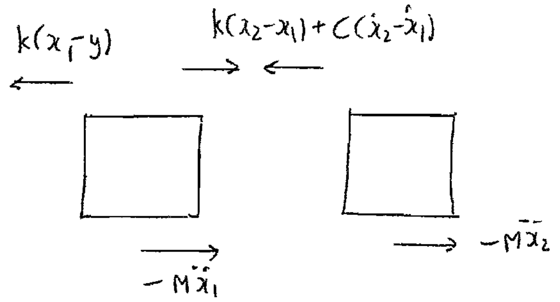
$\nu_0^+ = \dfrac{1}{2\pi}\left(\dfrac{\sigma_{\dot{S}}}{\sigma_S}\right) = \dfrac{1}{2\pi}\dfrac{7.9\times10^3}{5.04} = 250$ (ω_n in Hz)
$T = 5\times60$
$\sigma = 5.04$

$\Rightarrow D = \dfrac{250\times5\times60}{8\times10^7}\sqrt{\dfrac{\pi}{2}}\,3(5.04)^3 = \underline{0.4513}$ [35%]

c) (i) Note that σ_x^2 is proportional to ω_n^{-3}
$\Rightarrow \sigma_S^2$ is proportional to ω_n^{-3}
Also ν_0^+ is proportional to ω_n
$\Rightarrow D$ is proportional to $\omega_n\times(\omega_n^{-3})^{3/2} \Rightarrow D$ varies as $\underline{\alpha^{-7/2}}$ [20%]

(ii) Increasing ω_n increases ν_0^+ in proportion to ω_n but decreases b/σ_S in proportion to $\omega_n^{-3/2}$
$\Rightarrow \nu_b^+$ will decrease $\Rightarrow$ Probability of exceeding 25MPa will decrease [10%]

2 a)



$$\left.\begin{aligned} &M\ddot{x}_2+k(x_2-x_1)+c(\dot{x}_2-\dot{x}_1)=0 \\ &M\ddot{x}_1+k(x_1-y)-k(x_2-x_1) \\ &\qquad -c(\dot{x}_2-\dot{x}_1)=0 \end{aligned}\right\}$$

In Matrix form:
$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{pmatrix} c & -c \\ -c & c \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}+\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} ky \\ 0 \end{pmatrix}$$

For transfer functions, assume x_1 and x_2 are $\propto e^{i\omega t}$

$$\Rightarrow \begin{pmatrix} -M\omega^2+ci\omega+2k & -ci\omega-k \\ -ci\omega-k & -M\omega^2+ci\omega+k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} ky \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\frac{1}{\text{Det}}\begin{pmatrix} -M\omega^2+ci\omega+k & \checkmark \\ ci\omega+k & \checkmark \end{pmatrix}\begin{pmatrix} ky \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{x_1(\omega)=\frac{k(-M\omega^2+ci\omega+k)}{\text{Det}}\,y(\omega)}$$

$$\underline{x_2(\omega)=\frac{(ci\omega+k)k}{\text{Det}}\,y(\omega)}$$

$$\begin{aligned} \text{Det} &= (-M\omega^2+2k+ci\omega)(-M\omega^2+k+ci\omega)-(ci\omega+k)^2 \\ &= M^2\omega^4-M\omega^2(3k+2ci\omega)+(2k+ci\omega)(k+ci\omega)-(ci\omega+k)^2 \\ &= M^2\omega^4-3kM\omega^2+k^2+(-2M\omega^2 ci\omega+kci\omega) \end{aligned}$$

$$\underline{\text{Det}=M^2\omega^4-3kM\omega^2+k^2+ci\omega(-2M\omega^2+k)}$$ [60%]

(b) $F=k(x_2-x_1) \Rightarrow \underline{F(\omega)=\dfrac{k^2M\omega^2}{\text{Det}}\,y(\omega)}$ [From above results]

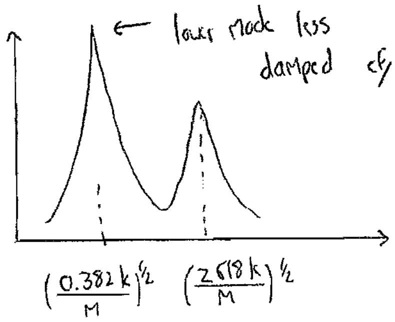
$$S_{FF}(\omega)=\left|\frac{F}{y}\right|^2 S_{yy}(\omega)=\underline{\frac{k^4M^2\omega^4}{|\text{Det}|^2}S_{yy}(\omega)}$$

b cont)

$$S_{FF}(\omega) = \left[\frac{k^4M^2\omega^4}{(M^2\omega^4 - 3kM\omega^2 + k^2)^2 + c^2\omega^2(2M\omega^2-k)^2}\right] S_{yy}(\omega)$$

This part is zero at resonant frequencies

$$\omega_n^2 = \frac{3kM \pm \sqrt{9k^2M^2 - 4M^2k^2}}{2M^2} = \left(\frac{k}{M}\right)\left[\underbrace{\frac{3\pm\sqrt{5}}{2}}_{0.382 \text{ and } 2.618}\right]$$



cf value of denominator at resonance

$= c^2\omega_n^2(2M\omega_n^2 - k)^2$

smaller for ω_1 than ω_2

[40%]

c) Spectrum of $\ddot{F}$ $= \omega^4 S_{FF}(\omega) = \dfrac{k^4M^2\omega^8}{|Det|^2} S_{yy}(\omega)$

$$\sigma_{\ddot{F}}^2 = \int_{-\infty}^{\infty} \omega^4 S_{FF}(\omega)\,d\omega$$

But, as $\omega \to \infty$, $\omega^4 S_{FF}(\omega) \to$ constant, since both numerator and denominator $\propto \omega^8$

$\Rightarrow$ $\underline{\sigma_{\ddot{F}}^2 = \infty}$

[20%]

Q.3
(i) Introduce a new variable y.

$$\dot{x} = y$$
$$\dot{y} = -x + \pi/_2 \sin x$$

Equilibrium points:

$\dot{x} = 0$ and $\dot{y} = 0$

$y = 0$ and $-x + \pi/_2 \sin x = 0$

$\dot{x} = 0$ and $x = \pi/_2 \sin x$

$\dot{x} = 0$ and $x = 0$, $x = \pm\pi/_2$. [30%]

(ii) Linearize about $\dot{x} = 0$, $x = 0$.

$\sin x \simeq x$

so matrix $A = \begin{bmatrix} 0 & 1 \\ \pi/_2 - 1 & 0 \end{bmatrix}$

solving for eigenvalues of A

$$\begin{vmatrix} -\lambda & 1 \\ \pi/_2 - 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\pi/_2 - 1) = 0$$

$$\lambda = \pm\sqrt{(\pi/2 - 1)}$$

Real and opposite sign $\Rightarrow$ saddlepoint

Now linearize about $\dot{x} = 0$, $x = \pi/2$

Make substitute $z = \pi/2 - x$

or $x = \pi/2 - z$

$\therefore$ $\sin x = \sin(\pi/2 - z)$

$= \cos z$

and $\cos z \simeq 1 - z^2/2$

$\dot{x} = y$ or $\dot{z} = -y$

$\therefore$ $\dot{y} = -(\pi/2 - z) + \pi/2(1 - z^2/2)$

$\simeq z$

and matrix A

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Characteristic equation:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\lambda^2 + 1 = 0$

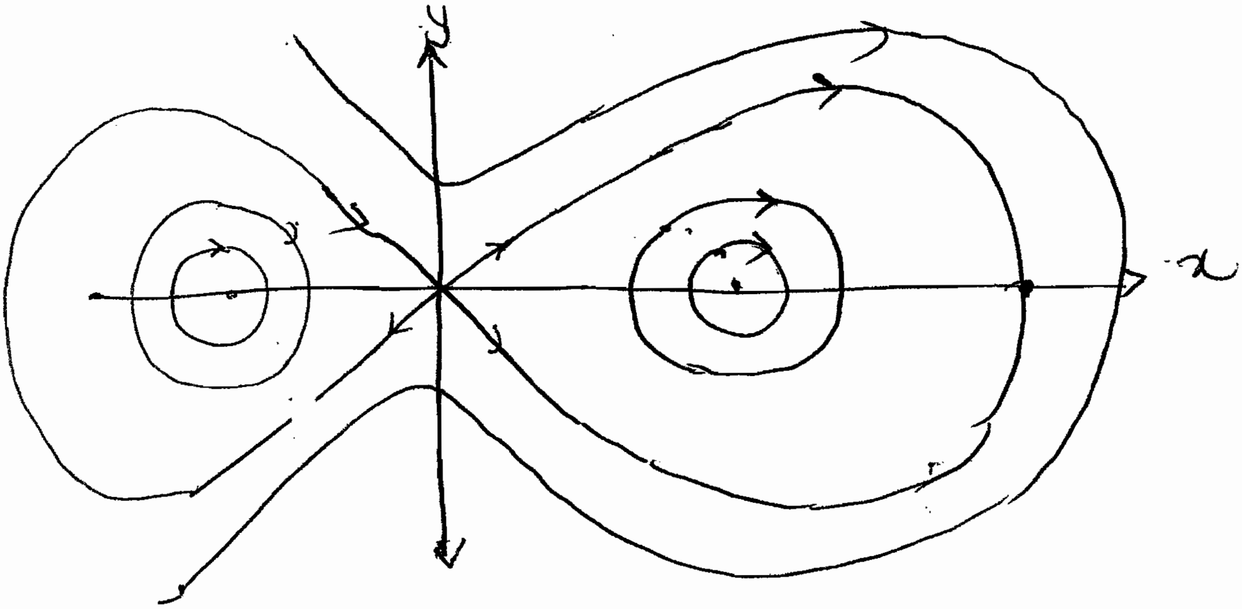
or $\lambda = \pm i$

[40%]

$\therefore$ The equilibrium point is a center

Same applies for $x = -\pi/2$

(iii)



[30%]

Q4

(i) $m\ddot{x} + \frac{2T}{\ell}x + \frac{AE}{\ell^3}x^3 = 0$

Rewrite as:

$$\ddot{x} + \omega_n^2 x + \mu x^3 = 0$$

where $\omega_n^2 = \frac{2T}{m\ell}$

and $\mu = \frac{AE}{m\ell^3}$

$$x = \underbrace{x_1(t)}_{\text{linear}} + \underbrace{\xi(t)}_{\text{non-linear term}}$$

$$(\ddot{x}_1 + \ddot{\xi}) + \omega_n^2(x_1 + \xi) + \mu(x_1 + \xi)^3 \approx 0$$

separate as:

$$\ddot{x}_1 + \omega_n^2 x_1 + \mu x_1^3 = 0 \quad \text{—(1)}$$

and $\ddot{\xi} + (\omega_n^2 + \mu 3x_1^2)\xi = 0$ —(2)

For the first equation: $x_1 \simeq a\sin\omega_n t$ (onset of nonlinearity)

Substitute into (2)

$$\ddot{\xi} + \left[\omega_n^2 + \mu 3(a^2\sin^2\omega_n t)\right]\xi = 0$$

$$\ddot{\xi} + \left[\left(\omega_n^2 + \frac{3\mu}{2}a^2\right) - \frac{3\mu}{2}a^2\cos 2\omega_n t\right]\xi = 0$$

which is the Mathieu equation [50%]

(ii) Method of iteration

$$\ddot{\xi}_1 + \omega_n'^2 \xi_1 = 0 \quad \text{where } \omega_n'^2 = \left(\omega_n^2 + \frac{3\mu}{2} a^2\right)$$

$$\xi_1 \simeq A \cos \omega_n' t$$

$$\ddot{\xi}_2 + \omega_n'^2 \xi_2 = \frac{3\mu}{2} a^2 A \cos 2\omega_n t \cos \omega_n' t$$

$$\ddot{\xi}_2 + \omega_n'^2 \xi_2 = \frac{3\mu A a^2}{2}\left[\cos(2\omega_n + \omega_n')t + \cos(2\omega_n - \omega_n')t\right]$$

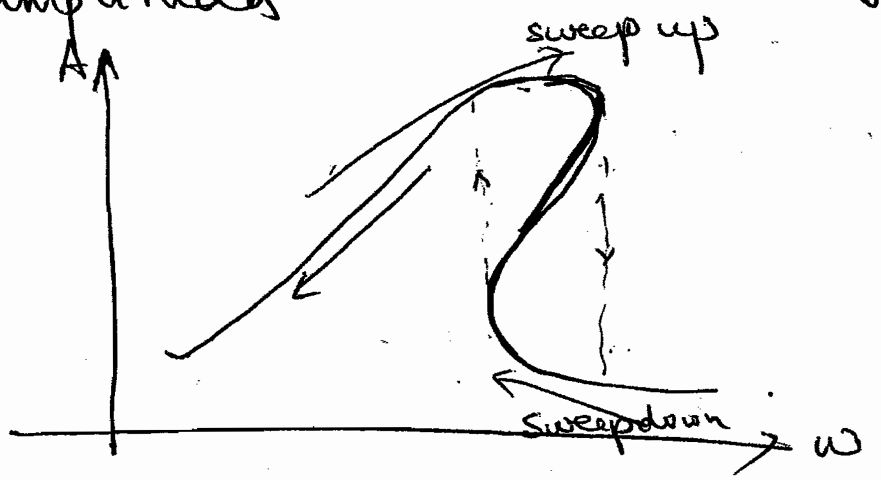
Assuming $(2\omega_n - \omega_n') \neq \omega_n'$ (true).

$$\xi_2 = \alpha \cos(\omega_n + \omega_n')t + \beta \cos(\omega_n - \omega_n')t$$

$$\alpha = \frac{3\mu A a^2}{2(\omega_n^2 - (\omega_n + \omega_n')^2)} \quad \& \quad \beta = \frac{-3\mu a^2 A}{2(\omega_n^2 - (\omega_n - \omega_n')^2)}$$

[40%]

(iii) The system would describe spring hardening behaviour of course with a jump phenomenon exhibited for high forcing amplitudes



[10%]