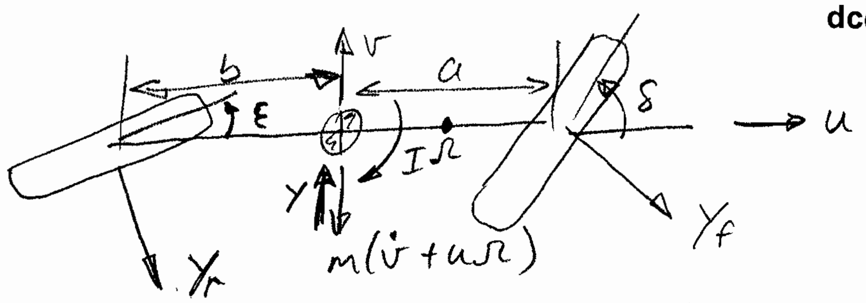
2007 IIB 4C8 Prof D Cebon
dc@eng.cam.ac.uk

1. (a)



The force generated by the rear tyre is

$$Y_r = C_r \alpha_r = C\left(\frac{v-b\Omega}{u} - \varepsilon\right) \quad — (1)$$

but $\varepsilon = \gamma Y_r$ so $Y_r(1+\gamma C) = C\left(\frac{v-b\Omega}{u}\right)$

$$\Rightarrow Y_r = \frac{C}{u}\frac{(v-b\Omega)}{(1+\gamma C)} \quad — (2)$$

& $Y_f = C_f \alpha_f = C\left(\frac{v+a\Omega}{u} - \delta\right)$, as usual — (3)

(b) Considering δ & ε to be small, lateral motion is

$$m(\dot{v}+u\Omega) + \frac{C}{u}\left(\frac{v-b\Omega}{1+\gamma C}\right) + C\left(\frac{v+a\Omega}{u} - \delta\right) - Y = 0 \quad (4)$$

& yaw motion is

$$I\dot{\Omega} + aC\left(\frac{v+a\Omega}{u} - \delta\right) - \frac{bC}{u}\left(\frac{v-b\Omega}{1+\gamma C}\right) = 0 \quad — (5)$$

In matrix form this is

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \dot{v} \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} \frac{C}{u}\left(\frac{1}{1+\gamma C}+1\right) & mu + \frac{C}{u}\left(a - \frac{b}{1+\gamma C}\right) \\ \frac{C}{u}\left(a - \frac{b}{1+\gamma C}\right) & \frac{C}{u}\left(a^2 + \frac{b^2}{1+\gamma C}\right) \end{bmatrix}\begin{Bmatrix} v \\ \Omega \end{Bmatrix} = \begin{Bmatrix} Y + C\delta \\ aC\delta \end{Bmatrix} \quad (6)$$

$$\Rightarrow [M]\,\dot{\underline{y}} + [K]\,\underline{y} = \underline{F}$$

(c) (i) For stability calculation, put $\underline{F} = 0$ and let $\underline{y} = \underline{Y}e^{\lambda t} \Rightarrow \dot{\underline{y}} = \lambda \underline{Y} e^{\lambda t}$

Then (6) becomes $([K] + \lambda[M])\underline{Y} = 0$

for which the stability condition is $|[K] + \lambda[M]| = 0$

Giving $a_2\lambda^2 + a_1\lambda + a_0 = 0$, for which the stability condition is that $a_0 > 0, a_1 > 0$ & $a_2 > 0$.

1 (c) (ii) When subjected to a steady side force, Y, the steady state motion will be $\dot{v} = \dot{\Omega} = 0$
So ⑥ can be written

$$[K]\underline{y}_{ss} = \underline{F} \quad , \text{where } \underline{F} = \begin{Bmatrix} Y \\ 0 \end{Bmatrix}$$

Invert this to give $\begin{Bmatrix} v_{ss} \\ \Omega_{ss} \end{Bmatrix} = [K]^{-1}\begin{Bmatrix} Y \\ 0 \end{Bmatrix}$

(iii) For a constant steer angle Δ, the steady motion will be circular with $\dot{v} = \dot{\Omega} = 0$

So $F = C\Delta\begin{Bmatrix} 1 \\ a \end{Bmatrix}$ and ⑥ can be written

$$[K]\,\underline{y}_{ss} = \underline{F}$$

Invert to give $\begin{Bmatrix} v_{ss} \\ \Omega_{ss} \end{Bmatrix} = [K]^{-1} C\Delta\begin{Bmatrix} 1 \\ a \end{Bmatrix}$

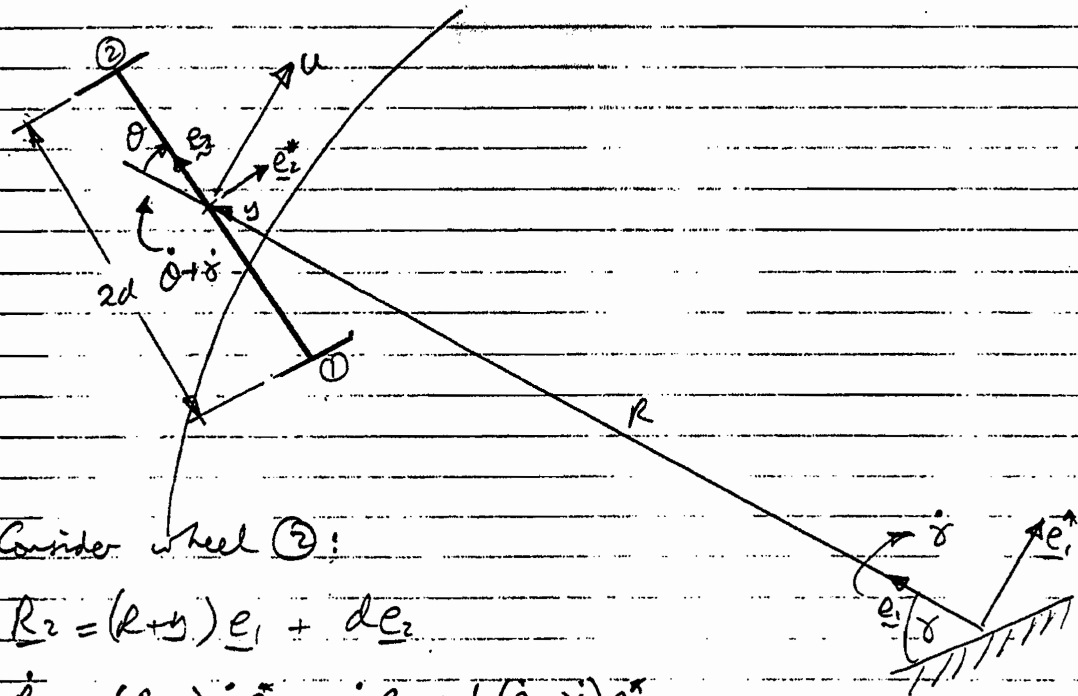
solve for Ω_{ss} then put $\Omega = \frac{U}{R}$ for circular motion

(iv) for $\delta = \Delta e^{i\omega t}$, the response will be $\underline{y} = \begin{Bmatrix} \bar{V} \\ \bar{\Omega} \end{Bmatrix} e^{i\omega t}$

So ⑥ becomes $\left(i\omega[M] + [K]\right)\underline{Y}\,\cancel{e^{i\omega t}} = C\Delta\begin{Bmatrix} 1 \\ a \end{Bmatrix}\cancel{e^{i\omega t}}$

Invert: $\underline{Y} = \begin{Bmatrix} \bar{V} \\ \bar{\Omega} \end{Bmatrix} = \left([K] + i\omega[M]\right)^{-1} C\Delta\begin{Bmatrix} 1 \\ a \end{Bmatrix}$

Q2



Consider wheel ②:

$$\underline{R}_2 = (R+y)\,\underline{e}_1 + d\,\underline{e}_2$$

$$\dot{\underline{R}}_2 = (R+y)\dot\gamma\,\underline{e}_1^* + \dot y\,\underline{e}_1 + d(\dot\theta+\dot\gamma)\,\underline{e}_2^*$$

Resolving: $e_1 \approx e_2 - \theta\, \underline{e}_2^*$ & $\underline{e}_1^* \approx e_2^* + \theta e_2$

Then putting $u = (R+y)\dot\gamma$ and neglecting the small $\dot y\theta$ term:

$$\dot{\underline{R}}_2 = [u + d(\dot\theta+\dot\gamma)]\,\underline{e}_2^* + [u\theta + \dot y]\,\underline{e}_2 \qquad (1)$$

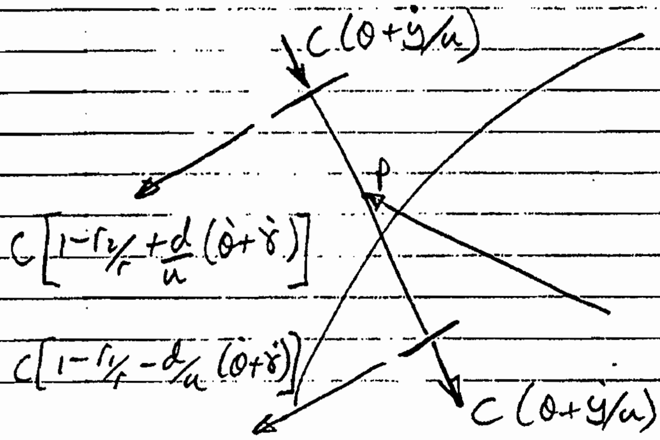
If the wheelset rotates with angular velocity $\omega = u/r$

the longitudinal creep velocity is $V_2 = u + d(\dot\theta+\dot\gamma) - \frac{r_2 u}{r}$ — (2)

Similarly for wheel ①.

Assuming the lateral & long'l creep forces are

$X = -C\frac{V_x}{u}$ & $Y = -C\frac{V_y}{u}$ gives: — (3)



Σ lateral forces $= 2C\left(\dot{y}/u + \theta\right)$ ——— (4)

Σ long'l forces $= 0$

Σ Moments at $P = dC\left[\left(\frac{r_2 - r_1}{r}\right) - \frac{2d}{u}\left(\dot{\theta} + \dot{\gamma}\right)\right]$

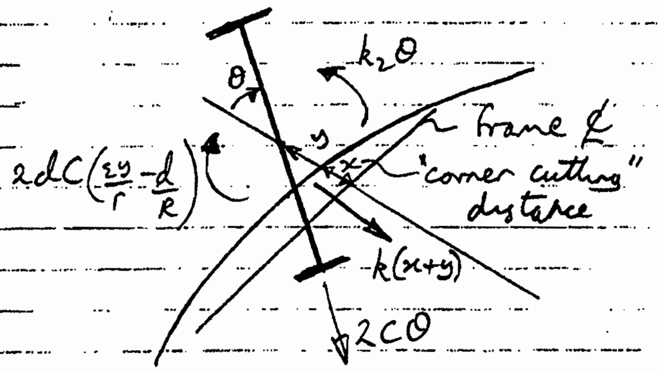
From wheel geometry, $r_2 - r_1 = 2\varepsilon y$; also $\dot{\gamma} = \frac{u}{R}$

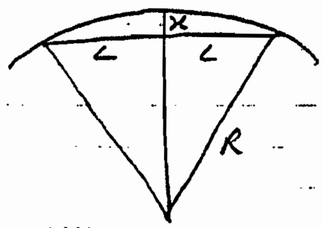
So $\Sigma M_P = 2dC\left[\frac{\varepsilon y}{r} - \frac{d\dot{\theta}}{u} - d/R\right]$ ——— (5)

Assumptions: $C_{11} = C_{22} = C$
Linear creep
No spin creep
No steer angle δ,
Large R, small y, θ

(b) Steady state turn with high damping $\Rightarrow \dot{y} = \dot{\theta} = 0$

The creep forces and moments are balanced by the springs





$$\left.\begin{aligned} \Sigma F: \quad & k_1(x+y) + 2C\theta = 0 \\ \Sigma M: \quad & k_2\theta - 2dC\left(\varepsilon y/r - d/R\right) = 0 \end{aligned}\right\} \text{(6)}$$

$$x = R - \sqrt{R^2 - L^2} \approx L^2/2R \quad \text{— (7)}$$

Combining (6) & (7) to eliminate x & θ:

$$k_1\left(L^2/2R + y\right) + 2C\left[\frac{2dC}{k_2}\left(\varepsilon y/r - d/R\right)\right] = 0$$

$$\Rightarrow y = \frac{\frac{4d^2C^2}{k_1k_2R} - \frac{L^2}{2R}}{1 + \frac{4dC^2}{k_1k_2}\frac{\varepsilon}{r}}$$

$y = 0$ if $k_1k_2 = 8d^2C^2/L^2$

Then (6a) $\rightarrow \theta = -\frac{k_1L^2}{4CR}$

3 a) For a mass m, $\underline{F} = m \nabla u$

So if $u = \frac{GM}{|r|}$, then $\underline{F} = m \frac{\partial u}{\partial r} \underline{e}_r$

$$= -m \frac{GM}{r^2} \underline{e}_r$$

Take a blob of matter, density ρ, volume V_0 at the centre of a sphere of radius a. Outwards force per unit mass at any point on the sphere's surface is

$$\underline{F} = \nabla u = -\frac{G \rho V_0}{a^2} \underline{e}_r$$

By Gauss, $\iiint_{\text{volume}} \nabla^2 u \, dv = \iint_{\text{surface}} \nabla u . \underline{ds}$

$$\therefore \iiint \nabla^2 u \, dv = -\frac{G \rho V_0}{a^2} \times 4\pi a^2$$

$$= -4\pi G \rho V_0$$

Independent of a, so shrink sphere down to size of blob, and get

$$\nabla^2 u \cancel{V_0} = -4\pi G \rho \cancel{V_0}$$

i.e. Poisson's equation

b) If $u = R(r) T(\theta)$, then

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial RT}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial RT}{\partial \theta}\right) = 0$$

$$2rT \frac{dR}{dr} + r^2 T \frac{d^2R}{dr^2} + \tan\theta R \frac{dT}{d\theta} + R \frac{d^2T}{d\theta} = 0$$

i.e. $$\frac{2r}{R} \frac{dR}{dr} + \frac{r^2}{R} \frac{d^2R}{dr^2} = -\frac{\tan\theta}{T} \frac{dT}{d\theta} - \frac{1}{T} \frac{d^2T}{d\theta^2}$$

3 b) cont.

LHS & RHS must both be equal to a constant - setting the constant to $n(n+1)$ yields the equation shown.

For r, assume $R = Ar^p$

then $p(p-1)r^p + 2pr^p - n(n+1)r^p = 0$

i.e. $p^2 + p - n(n+1) = 0$

or $(p-n)(p+(n+1)) = 0$

Whence $R = Ar^n + Br^{-n-1}$

ignore, because $\to \infty$ as $r \to \infty$

(c) Second equation is Legendre's equation, which is solved by Legendre polynomials $P_n(\cos\theta)$

Solution for $n=0$ is point mass solution i.e. $U = \mu/r$

Solution for $n=1$ is eliminated by putting the origin at the Earth's C of G

Then $U = \mu/r\left(1 - \sum_{n=2}^{\infty}\left(\frac{R_E}{r}\right)^n J_n P_n(\cos\theta)\right)$

The $n=2$ term models the equatorial 'bulge' of the earth

3 c) cont.

We can ignore $U = f(\phi)$ because the effects are small compared to the J_2 term, and are also 'smeared out' because the earth is spinning beneath the satellite.

d) Two effects:

i) Plane of orbit will precess about the earth's axis, because of the couple applied to the orbit by the earth's equatorial bulge. This will cause the 'Line of Nodes' to change slowly with time.

ii) The orbital 'ellipse' will slowly rotate in the plane of the orbit, because it is not a Keplerian orbit (earth does not behave as a point mass). This will cause the 'Argument of Perigee' to change slowly with time.

4 (a) From given equations, $\dot{\theta} = h/r^2$

so $\ddot{r} - h^2/r^3 = -\mu/r^2$

Let $u = 1/r$

Then $\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{d(1/u)}{d\theta} \times \frac{h}{r^2}$

$= hu^2\left(-\frac{1}{u^2}\right)\frac{du}{d\theta} = -h\frac{du}{d\theta}$

So $\ddot{r} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2} \times hu^2$

$= -h^2u^2\frac{d^2u}{d\theta^2}$

$\therefore -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2$

or $\frac{d^2u}{d\theta^2} + u = \mu/h^2$

h is specific moment of momentum

and μ is GM for gravitational problems

(b) Solution is $u = \mu/h^2 + A\cos(\theta - \theta_0)$

or $u = \mu/h^2(1 + e\cos\theta)$

whence $r = \frac{h^2/\mu}{1 + e\cos\theta}$

if $\theta = 0$ @ perigee

4 (cont)

From data sheet, substitute

$$h^2/\mu = L = a(1-e^2)$$

to give $$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

i.e. an ellipse with semi-major axis a and eccentricity e

(c) Area of ellipse $= \pi a b$

but $b = \sqrt{aL}$ from sheet,

so area $= \pi a^{3/2} h/\sqrt{\mu}$

Rate of sweep is $h/2$,

so period $= \dfrac{2\pi a^{3/2}}{\sqrt{\mu}}$

which proves Kepler's 3rd Law.