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Part IIB 2007 Module 4C9 Solutions

1(a)

$$a_i = e_{ijk} b_j c_k \qquad b_i = e_{ijk} g_j h_k$$

Substituting

$$a_i = e_{ijk} e_{jlm} g_l h_m c_k$$

But (data sheet)

$$e_{ijk} e_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$a_i = - e_{jik} e_{jlm} g_l h_m c_k$$

$$\text{ie. } a_j = - e_{ijk} e_{ilm} g_l h_m c_k$$

$$= - \delta_{jl}\delta_{km} g_l h_m c_k + \delta_{jm}\delta_{kl} g_l h_m c_k$$

$$\Rightarrow - g_j h_k c_k + g_k h_j c_k$$

$$a_j = g_k c_k h_j - h_k c_k g_j$$

$$\text{ie. } a_i \Rightarrow g_k c_k h_i - h_k c_k g_i$$

$$g_k c_k = \underline{g} \cdot \underline{c} \qquad h_k c_k = \underline{h} \cdot \underline{c}$$

$$\therefore \underline{a} = (\underline{g} \cdot \underline{c})\underline{h} - (\underline{h} \cdot \underline{c})\underline{g}$$

In vectors $\underline{a} = \underline{b} \times \underline{c}$; $\underline{b} = \underline{g} \times \underline{h}$

$$\therefore \underline{a} = (\underline{g} \times \underline{h}) \times \underline{c}$$

vector triple product (Data book) $\Rightarrow (\underline{g} \cdot \underline{c})\underline{h} - (\underline{h} \cdot \underline{c})\underline{g}$

1. (b)

(i) For an elastic-perfectly plastic solid, as the load acting on it is gradually increased, we expect a limit value of the load to be reached (or approached in an asymptotically manner) corresponding to unrestrained plastic deformation with no change in stresses. The stress state at the limit load is defined as the limit stress state.

(ii) Assume we can find $\dot{\sigma}_{ij} \neq 0$ at the limit stress state, with $\dot{\sigma}_{ij} n_j = 0$ on S_T and associated with $\dot{\varepsilon}_{ij}$ and $\dot{u}_i$ such that $\dot{u}_i = 0$ on S_u ($S = S_T + S_u$). Then note

$$0 = \int_S \dot{\sigma}_{ij} n_j \dot{u}_i \, dS \underset{\uparrow}{=} \int_V \dot{\sigma}_{ij} \dot{\varepsilon}_{ij} \, dV$$

apply principle of virtual work (PVW)

But $\dot{\sigma}_{ij} \dot{\varepsilon}_{ij} > 0$ (Drucker's postulate), unless $\dot{\sigma}_{ij} = 0$ everywhere in V. By contradiction, $\dot{\sigma}_{ij}$ must vanish. #

23. (a) Using J_2-flow theory, one has

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\, s_{ij}$$

In simple tension, $\sigma = \sigma_y$ at yielding, and

$$J_2 = \frac{1}{2} s_{ij} s_{ij} = \frac{\sigma_y^2}{3}, \quad s_{11} = \frac{2}{3}\sigma_y$$

$$\Rightarrow \dot{\varepsilon}^{p} = \dot{\lambda}\,\frac{2}{3}\sigma_y \qquad \Rightarrow \dot{\lambda} = \frac{3}{2}\frac{\dot{\varepsilon}^{p}}{\sigma_y}$$

Generalize this to 3-D :

Mises criterion

$$\dot{\lambda} = \frac{3}{2}\frac{\dot{\varepsilon}_e^{p}}{\sigma_e} \qquad (\text{ Note } \sigma_e = \sigma_y \text{ at yielding}$$

$$\Rightarrow \quad \dot{\varepsilon}_{ij}^{p} = \frac{3}{2}\frac{\dot{\varepsilon}_e^{p}}{\sigma_e} s_{ij}$$

(b) $\because$ No elasticity $\therefore$ $\dot{\varepsilon}_r = \dot{\varepsilon}_r^{p}$

Also, $\sigma_r = 0$, $\sigma_\theta = \dfrac{p r_0}{t_0}$, $\sigma_z = \dfrac{\sigma_\theta}{2}$ (close-ended cylinder

$$\Rightarrow \sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2\right]} = \frac{\sqrt{3}}{2}\sigma_\theta$$

$$\Rightarrow \dot{\varepsilon}_r = \dot{\varepsilon}_r^{p} = \frac{3}{2}\frac{\dot{\varepsilon}_e^{p}}{\sigma_e}\left[\sigma_r - \frac{1}{3}(\sigma_\theta + \sigma_z)\right]$$

$$= -\frac{3}{2}\frac{\dot{\varepsilon}_e^{p}}{\sigma_e}\cdot\frac{1}{3}\cdot\frac{3}{2}\sigma_\theta$$

$$= -\frac{3}{4}\frac{\dot{\varepsilon}_e^{p}}{\sigma_e}\cdot\frac{2}{\sqrt{3}}\sigma_e$$

$$\Rightarrow \quad d\varepsilon_r = -\frac{\sqrt{3}}{2}\, d\varepsilon_e^{p}$$

Integration gives

$$\varepsilon_r = -\frac{\sqrt{3}}{2}\varepsilon_e^{p}$$

since $\varepsilon_r = \varepsilon_e^{p} = 0$ when $p = 0$.

2 题. (c) $$d\varepsilon_r = \frac{dt}{t}$$

$$\Rightarrow \quad \varepsilon_r = \ln(t/t_0)$$

Substitution of the above into (b) gives

$$t = t_0 \exp\left(-\frac{\sqrt{3}}{2}\varepsilon_e^p\right)$$

(d) $$\sigma_e = \frac{\sqrt{3}}{2}\sigma_\theta = \frac{\sqrt{3}}{2}\frac{p r_0}{t_0}$$

$$= \frac{\sqrt{3}}{2} \times \frac{10 \times 200}{4} = 250\sqrt{3} \text{ MPa}$$

Generalize $\sigma = A\varepsilon^n$ to 3-D :

$$\sigma_e = A(\varepsilon_e^p)^n \quad \Rightarrow \quad \varepsilon_e^p = \left(\frac{\sigma_e}{800}\right)^4$$

$$\Rightarrow \quad t = t_0 \exp\left\{-\frac{\sqrt{3}}{2}\left(\frac{250\sqrt{3}}{800}\right)^4\right\}$$

$$= 3.714 \text{ mm}$$

Thus, the decrease in wall thickness is

$$\Delta t = t_0 - t = 0.286 \text{ mm}$$

3 (a) $u_r = \frac{3Bk}{h^2}(h-z)rz \qquad u_z = -Bk\left(\frac{3z^2}{h} - \frac{2z^3}{h^2}\right) - Bz$

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{3Bk(h-z)z}{h^2} \qquad \varepsilon_z = \frac{\partial u_z}{\partial z} = -Bk\left(\frac{6z}{h} - \frac{6z^2}{h^2}\right) - B$$

$$\varepsilon_\theta = \frac{u}{r} = \frac{3Bk(h-z)z}{h^2}$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} = \frac{3Bk}{h^2}(h-2z)$$

$$\therefore e = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{3Bk(h-z)z}{h^2} + \frac{3Bk(h-z)z}{h^2} - Bk\left(\frac{6z}{h} - \frac{6z^2}{h^2}\right) - B$$

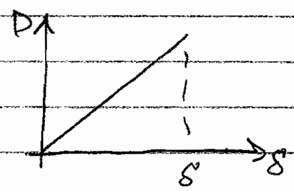
i.e. $e = -B$

So material is compressible unless $B = 0$, so $\nu \neq 0$

(b) If $u_z = -\delta$ at $z = h$, then

$$-\delta = -Bk(3h - 2h) - Bh$$

i.e. $\delta = Bh(k+1)$



Mat^l is linear elastic

So $V = \frac{1}{2} P.\delta$

Mean Axial stress $= P/\pi a^2$

| —— strain | $= \delta/h$

$$\therefore E_a = \frac{P}{\pi a^2}\frac{h}{\delta}$$

$$\therefore V = \frac{1}{2} \cdot \frac{\pi a^2 \delta \cdot E_a}{h} \cdot \delta$$

$$V \Rightarrow \frac{\pi}{2} E_a \frac{a^2\delta^2}{h}$$

(c) Now also $V = 2\mu\pi a^2 h B^2 [\;]$

equating $2\mu\cancel{\pi}\cancel{a^2} h B^2 [\;] = \frac{\cancel{\pi}}{2} E_a \frac{\cancel{a^2}\delta^2}{h}$

But $\mu = \frac{E}{2(1+\nu)}$

$$\therefore \frac{E}{1+\nu} h B^2 \left[\;\right] = \frac{E_a}{2} \frac{\delta^2}{h}$$

But from (b) $\delta = Bh(k+1)$

$$\therefore \frac{E}{(1+\nu)} \cancel{h}\cancel{B^2} [\;] = \frac{E_a}{2} \cancel{B^2}\cancel{h}(k+1)^2$$

E_a is an Upper Bound. $\therefore \underbrace{\frac{E_a}{E} = \frac{2}{1+\nu} \cdot \frac{1}{(1+k)^2} \left[\frac{q+1}{2} + 2k^2 + k \right]}$

Minimising this expression is not straightforward but

$$\left.\frac{E_a}{E}\right)_{min} = \frac{1}{1+\nu} \left[\frac{2z(q+1)-1}{2z+q-1}\right] \qquad z = \frac{9}{10} + \frac{3}{8}(a/h)^2$$

$$\text{ie} \simeq \frac{1}{1+\nu}\left[\frac{2zq}{2z+q}\right] \qquad \& \quad q = \frac{1}{1-2\nu}$$

Now if $\nu \Rightarrow 0.5$ then $q \gg z$

$$\therefore \frac{E_a}{E} \simeq \frac{2}{3}\left\{\frac{2zq}{q}\right\} \Rightarrow \frac{2}{3} \cdot 2z$$

$$\therefore \frac{E_a}{E} \simeq \frac{4}{3}\left\{\frac{9}{10} + \frac{3}{8}(a/h)^2\right\}$$

$$= \underbrace{\frac{6}{5} + \frac{1}{2}(a/h)^2}$$

Curious - but no error - that when (a/h) small this tends to $6/5$ and not unity.

JAW - May 07