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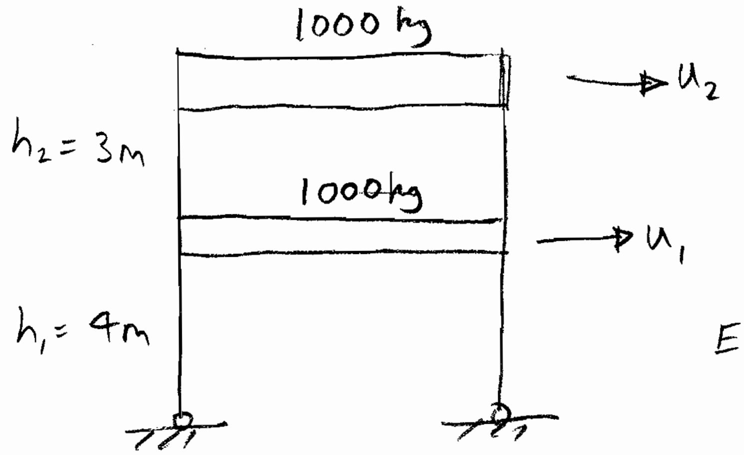


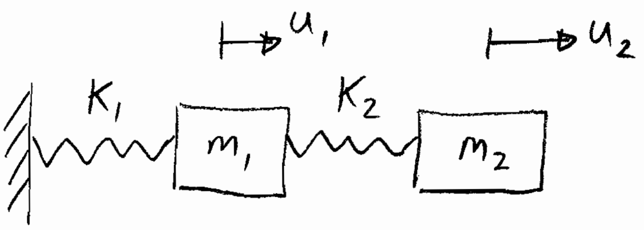
$m_2 = 1000$ kg

$m_1 = 1000$ kg

$EI = 8640$ kNm^2

$$K_1 = 2 \times \frac{3EI}{h_1^3} = \frac{2 \times 3 \times 8640}{4^3} = 810 \text{ kN/m}$$

$$K_2 = 2 \times \frac{12EI}{h_2^3} = \frac{2 \times 12 \times 8640}{3^3} = 7680 \text{ kN/m}$$



Eqns. of motion:

$$m_1 \ddot{u}_1 + K_1 u_1 + k_2 (u_1 - u_2) =$$

$$m_2 \ddot{u}_2 + K_2 (u_2 - u_1) = 0$$

In matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 8490 & -7680 \\ -7680 & 7680 \end{bmatrix} 10^3 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $u_1 = U_1 \sin\omega t \qquad u_2 = U_2 \sin\omega t$

$\therefore \quad \ddot{u}_1 = -\omega^2 U_1 \sin\omega t \qquad \ddot{u}_2 = -\omega^2 U_2 \sin\omega t$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times -\omega^2 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} 8490 & -7680 \\ -7680 & 7680 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 8490 - \omega^2 & -7680 \\ -7680 & 7680 - \omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(8490 - \omega^2)(7680 - \omega^2) - 7680^2 = 0$$

$$\omega^4 - 16170\omega^2 + 6220800 = 0$$

$$\therefore \quad \omega^2 = \left(16170 \pm \sqrt{16170^2 - 4 \times 6220800}\right)/2$$

$$\omega^2 = 394.3, \qquad 15775.7 \ (\text{rad/s})^2$$

$$\omega_1 = 19.86 \text{ rad/s} \qquad \omega_2 = 125.6 \text{ rad/s}$$
$$= 3.16 \text{ Hz} \qquad\qquad = 19.99 \text{ Hz}$$

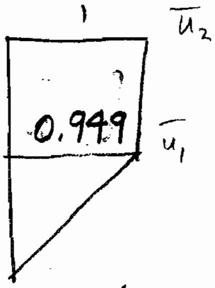
For ω_1 $\quad 8095.7 U_1 - 7680 U_2 = 0 \quad \therefore \dfrac{U_1}{U_2} = 0.94$

For ω_2 $\quad -7285.7 U_1 - 7680 U_2 = 0 \quad \therefore \dfrac{U_1}{U_2} = -1.0$

-7285.7

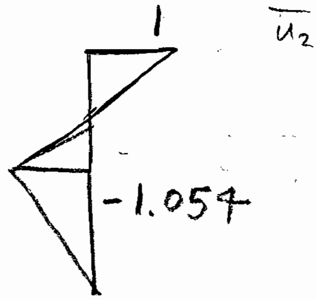
Mode Shapes

ω_1



$T_1 = 0.316\ s$

ω_2



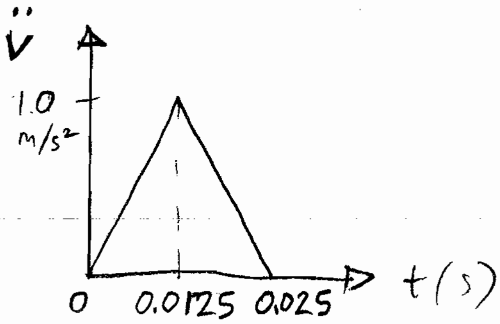
$T_2 = 0.05\ s$

$M_{eq1} = 1000 + 1000 \times (0.949)^2 = 1901\ kg$

$M_{eq2} = 1000 + 1000 \times (1.054$

$= 2111\ kg$

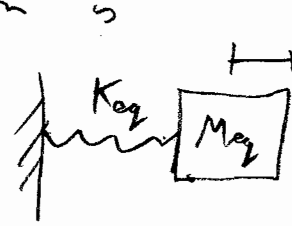
Earthquake :



For ω_1 $\quad \dfrac{t_d}{T_1} = \dfrac{0.025}{0.316} = 0.08 \quad \therefore \quad DAF = 0.2$

For ω_2 $\quad \dfrac{t_d}{T_2} = \dfrac{0.025}{0.05} = 0.5 \quad \therefore \quad DAF = 1.3$

Equiv SDOF system for each mode for ground vibration is



$\longrightarrow F_{eq} = -\Gamma M_{eq} \ddot{V}$

$K_{eq1} = 810 \times 0.949^2 + 7680 \times (1 - 0.949)^2 = 749.5\ kN/m$

$K_{eq2} = 810 \times (-1.054)^2 + 7680 (1 - -1.054)^2 = 33301\ kN/m$

$$\Gamma_1 = \frac{M_1\bar{U}_1 + M_2\bar{U}_2}{M_1\bar{U}_1^2 + M_2\bar{U}_2^2} = \frac{1000\times 0.949 + 1000\times 1}{1901}$$

$$= 1.025$$

$$\Gamma_2 = \frac{1000\times -1.054 + 1000\times 1}{2111} = -0.0256$$

$$F_{eq1} = -\Gamma_1 M_{eq1}\ddot{v} = -1.025\times 1901\times 10 = -19485\ N$$

$$F_{eq2} = -\Gamma_2 M_{eq2}\ddot{v} = 0.0256\times 2111\times 10 = 540.4\,N$$

Dynamic displacement:

Mode 1 $\quad \delta_{1dyn} = DAF_1 \times \dfrac{F_{eq1}}{K_{eq1}} = 0.2\times \dfrac{-19485}{749.5\times 10^3}$

$$= -5.2\times 10^{-3}\,m$$

Mode 2 $\quad \delta_{2dyn} = 1.3\times \dfrac{540.4}{33301\times 10^3} = 0.021\times 10^{-3}\,m$

Could use SRSS method i.e. $\delta = \sqrt{\delta_{1dyn}^2 + \delta_{2dyn}^2}$. However despite the high DAF for this period of loading Mode 2 is very weakly excited by ground motion and therefore the response is dominated by Mode 1.

2 a) The energy method or Rayleigh's method involves calculating the kinetic energy and potential energy of the structure when it is vibrating in a given mode shape. By equating the KE and PE the natural frequency of the structure can be obtained.

Several mode shapes can be assumed, each giving a corresponding natural frequency. The lowest of the estimated natural frequencies will be either equal to or greater than the true natural frequency of the structure.

The procedure involves seperating the time dependent response from the mode shape. For a distributed mass system, the mode shape is first assumed and the equivalent mass and stiffness are calculated as

$$M_{eq} = \int_0^L m \bar{u}^2 dx \quad \text{and} \quad K_{eq} = \int_0^L EI \left(\frac{d^2\bar{u}}{dx^2}\right)^2 dx .$$

Then $\omega = \sqrt{\dfrac{K_{eq}}{M_{eq}}} \geq \omega_n$ by Rayleigh's principle. (True natural frequency)

[20%]

2b) Consider first mode: $\bar{U}_1 = 1 - \cos\frac{\pi x}{20}$ as $n = 1$.

$L = 10\text{ m}$ $\quad EI = 8 \times 10^5 \text{ Nm}^2$ $\quad m = 300 \text{ kg/m}$.

$$\bar{U}_1 = 1 - \cos\frac{\pi x}{2L}$$

$$M_{1eq} = \int_0^L m\left(1 - \cos\frac{\pi x}{2L}\right)^2 dx .$$

$$= m\int_0^L \left[1 + \cos^2\frac{\pi x}{2L} - 2\cos\frac{\pi x}{2L}\right] dx .$$

$$= m\left[\int_0^L \cos^2\frac{\pi x}{2L}dx + \left[x - 2\cdot\frac{2L}{\pi}\sin\frac{\pi x}{2L}\right]_0^L\right] = m\left[L - \frac{4L}{\pi} + \frac{1}{2}\int_0^L\left(\cos\frac{\pi x}{10} + 1\right)dx\right]$$

$$= m\left[L - \frac{4L}{\pi} + \frac{L}{2}\right] = mL\left[\frac{3}{2} - \frac{4}{\pi}\right] = \underline{\underline{0.2267\ mL}}$$

$$M_{1eq} = 0.2267 \times 300 \times 10 = 680.28 \text{ kg}.$$

$$K_1 eq = \int_0^L EI \left(\frac{d^2\bar{U}_1}{dx^2}\right)^2 dx .$$

$$U_1 = 1 - \cos\frac{\pi x}{2L} \qquad \frac{d^2\bar{U}}{d\bar{x}^2} = \left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L} .$$

$$K_1 eq = EI\left(\frac{\pi}{2L}\right)^4 \int_0^L \cos^2\frac{\pi x}{2L}\, dx .$$

$$= EI\left(\frac{\pi}{2L}\right)^4 \times \int_0^L \frac{1}{2}\left(\cos\frac{\pi x}{L} + 1\right) dx .$$

$$= EI\frac{\pi^4}{(2L)^4} \times \frac{L}{2} = \frac{EI\,\pi^4}{32\,L^3} = \frac{8\times10^5 \times \pi^4}{32\times10^3} = 2435.23 \text{ N/m}$$

$$\therefore \quad \omega_1 = \sqrt{\frac{K_1 eq}{M_1 eq}} = \sqrt{\frac{2435.23}{680.28}} = 1.892 \text{ rad/s}$$

$$f_1 = 0.3 \text{ Hz} .$$

Consider 2^{nd} mode: $\bar{U}_2 = 1 - \cos\frac{2\pi x}{20} = 1 - \cos\frac{\pi x}{10} = 1 - \cos\frac{\pi x}{L}$

$$\therefore M_2 eq = \int_0^L m\left(1 - \cos\frac{\pi x}{L}\right)^2 dx = m\int_0^L \left(1 + \cos^2\frac{\pi x}{L} - 2\cos\frac{\pi x}{L}\right) dx$$

$$= m\left[L - 2\frac{L}{\pi}\sin\frac{\pi x}{L}\right]_0^L + m\int_0^L \cos^2\frac{\pi x}{L}\, dx .$$

$$= mL + m\left[\int_0^L \frac{1}{2}\left(\cos\frac{2\pi x}{L} + 1\right) dx\right.$$

$$= mL + \frac{m}{2}L + m\frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big]_0^L = \frac{3mL}{2} = \underline{\underline{4500}} \text{ kg} ,$$

$$K_2 eq = \int_0^L EI\left(\frac{d^2\bar{U}_2}{dx^2}\right)^2 dx \qquad \bar{U}_2 = 1 - \cos\frac{\pi x}{L} \qquad \frac{d^2\bar{U}_2}{dx^2} = \frac{\pi^2}{L^2}\cos\frac{\pi x}{L}$$

$$K_2 eq = EI\frac{\pi^4}{L^4}\int_0^L \cos^2\frac{\pi x}{L}\, dx = EI\frac{\pi^4}{L^4}\left[\frac{1}{2}\int_0^L\left(\cos\frac{2\pi x}{L} + 1\right) dx\right]$$

$$= \frac{EI\pi^4}{L^4} \times \frac{L}{2} + \underbrace{\frac{EI\pi^4}{L^4} \times \frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big]_0^L}_{0} = \frac{EI\pi^4}{2L^3}$$

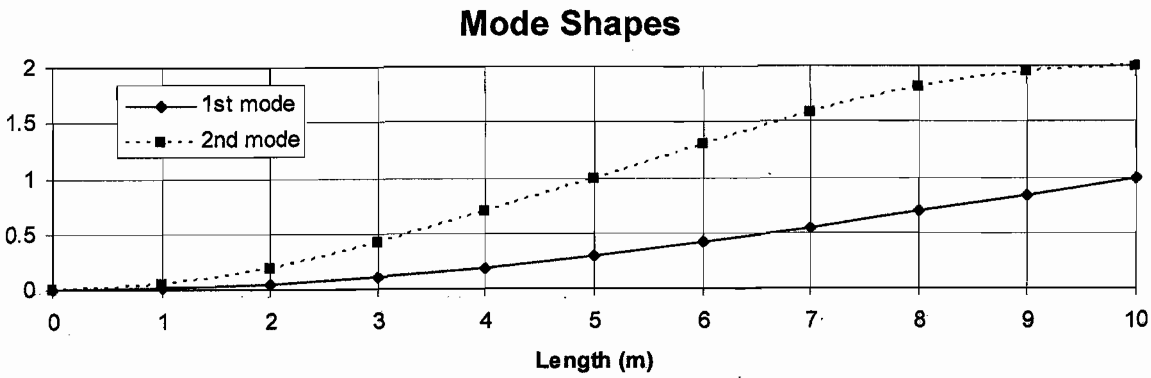
$$\therefore K_2 eq = \frac{8\times10^5 \times \pi^4}{2\times10^3} = 38963.64 \text{ N/m} .$$

$$\omega_2 = \sqrt{\frac{K_{2eq}}{M_{2eq}}} = \sqrt{\frac{38963.64}{450}} = 2.9425 \text{ rad/s}$$

$$f_2 = \underline{\underline{0.4683 \text{ Hz}}}.$$

So the first & 2nd mode natural frequencies are 0.3 Hz and 0.4683 Hz respectively. [40%]

2 c) The 1st mode shape is reasonable - but not the second mode shape. So it is important to check the mode shapes. Use 4th order polynomial to obtain correct mode shapes.



Note: As these are arbitrary mode shapes, the ordinate can be scaled to give unity at the free end of the beam. [20%]

2d) Determine the response of the beam when it is vibrating in the 1st mode and 2nd mode plotted above for any given time varying loading.

The mode superposition method involves superposing the response of the beam in the two modes shown above. Three possible ways for this superposition are

i) superpose the time histories of the response — This will be correct but time consuming.

ii) superpose maximum response in each mode — Easy but can be quite conservative — as maxima may not occur at the same time.

iii) SRSS Method — Square root of sum of squares of max response.

The SRSS method gives good results when modes are well seperated. Mode superposition gives good results only for orthogonal (eigen) modes. But real structures may not vibrate in eigen modes especially if they are non-linear systems. [20%]

3 a) An earthquake response spectrum is constructed using the actual time history recorded by an accelerometer/seismometer. Thus historic earthquakes such as El Centro earthquake motion are used to construct the response spectrum. by plotting the response of SDOF systems of different natural frequencies to the real earthquake input motion.

A design spectrum is obtained for any given region by considering many earthquake response spectra from historic earthquakes that have occurred in that region. Averaging and removal of or attenuation of high frequency components is usually carried out in a design spectrum. Similarly the max. response may be capped based on the knowledge of damping and soil characteristics in the given region. [15%]

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b) Loose sands have relatively large voids between the grains. When sheared such soils tend to suffer volumetric contraction. However if the soils are fully saturated, the volumetric contraction cannot take place on shearing, as the water in the voids needs to escape. Earthquake loading is far too rapid ~~for~~ to allow the pore fluid enough time to escape. Under these conditions, the pore water pressure increases instead of the reduction of volume of the saturated sand. As the soil stiffness and strength depend on effective stress $\sigma' = \sigma - u$.

As u increases σ' decreases causing the soil lose its strength. This is termed as soil liquefaction [15%]

3c) Shear wave velocity $v_s = \sqrt{\frac{G}{\rho}}$.

Unit weight $\gamma = 14.8$ kN/m³.

$\therefore$ Dry density $\rho = 1480$ kg/m³.

$v_s = 180$ m/s

$\therefore$ $G = \rho v_s^2 = 1480 \times 180^2 = 47.952$ MPa

Wolf's formula from Data sheets for horizontal Stiffness

$$K_{hx} = \frac{Gb}{2-\nu}\left[6.8\,(l/b)^{0.65} + 2.4\right] \text{ if } e = 0.$$

$2l = 3m$; $2b = 3m$; $e = 0$ $\quad \nu = 0.3$ for Sand

$\therefore$ $l = 1.5$ $\quad b = 1.5$ $\quad l/b = 1$.

$$K_{hx} = \frac{47.952 \times 1.5}{(2-0.3)} \times [6.8 + 2.4] = 389.25 \times 10^6 \text{ N/m}.$$

$$\omega_n = 2\pi f_n = 2\pi \times 20 = 40\pi. = \sqrt{\frac{K_{hx}}{M_{hx}}} \text{ rad/s}$$

$$\therefore M_{hx} = \frac{K_{hx}}{(40\pi)^2} = \frac{389.25 \times 10^6}{(40\pi)^2} = 24650 \text{ Kg}.$$

This includes bunker mass + Soil mass.

$\therefore$ Mass of Soil participating in horizontal vibration

$= 24650 - 6000 = 18650$ kg.

(approximately 3 times mass of bunker). [30%]

3d) During earthquake event; f_n dropped to 5 Hz.

$\therefore \omega_n = 2\pi f_n = 2\pi \times 5 = 10\pi$ rad/s.

$$\sqrt{\frac{K_{hx}}{M_{hx}}} = 10\pi. \Rightarrow K_{hx} = (10\pi)^2 M_{hx}$$

$$K_{hx} = 24650 \times 100 \times \pi^2$$

$$= 24.328 \times 10^6 \text{ N/m}.$$

3d) ∴ Reduction in horizontal stiffness

$$= 389.25 \times 10^6 - 24.328 \times 10^6$$

$$= 364.922 \times 10^6 \text{ N/m}$$

% Drop in stiffness = $\underline{\underline{93.75\ \%}}$ [20%]

3 e) Clearly the horizontal stiffness has reduced very significantly. This suggests that the foundation soil has suffered nearly "full liquefaction", This can lead severe settlement / rotation of the bunker.

In order to avoid damage to bunker either

i) provide vertical stone columns in the foundation soil below the bunker. - These will releive the excess pore water pressures quickly. or

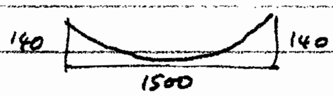
ii) Densify the ground prior to construction of the bunker. Vibro-densification ~~of~~ or similar methods of densification can be employed to reduce the liquefaction potential of the ground.

[20%]

Q4. Main span = 1.5 km

Towers 140 m above deck.

24 m wide, 2.9 m deep. $\cancel{28}$ 28.3 t/m.



EI

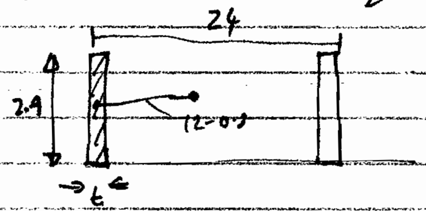
Check my values are reasonable:

say 7950 kg/m³ for steel

~~Is~~ say all steel

$\therefore \text{CSA} = \dfrac{28.3 \text{ t/m}}{7.95 \text{ t/m}^3} = 3.56 \text{ m}^2$

Say half each side $= \dfrac{3.56}{2} = 1.78 \text{ m}^2$



$t = \dfrac{1.78}{2.9} = 0.61 \text{m}$ $\qquad I_{max} = 1.78\text{m}^2 \times 2 \times (12-0.3)^2$

$= 487 \text{ m}^4$. Max Possible

$EI_{given} = \cancel{830}\ 44 \times 10^{12} \text{ N m}^2$ $\qquad E = 205 \times 10^9 \text{ N/m}^2$

$I_{given} = \dfrac{\cancel{830}\ 44 \times 10^{12}}{205 \times 10^9} = \cancel{4092 \text{ m}^4}\ \cancel{312}\ 214 \text{ m}^4$

which is less than max. possible

students do not need to do this

Estimate natural frequency

via a) Rayleigh Ritz

~~Central deflection of s/s span under UDL w per m length~~ $= \cancel{\dfrac{5wL^4}{384EI}}$

$$\omega^2 = \frac{\int_0^L EI (w'')^2 \, dx}{\int \rho A w^2 \, dx} \qquad \text{let } w = \sin\left(\frac{\pi x}{L}\right)$$

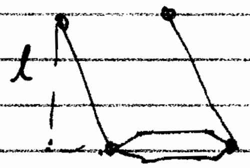
$$= EI \int_0^L \left[\frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)\right]^2 dx \Big/ \rho A \int_0^L \left[\sin\left(\frac{\pi x}{L}\right)\right]^2 dx$$

$$= \frac{\pi^4}{L^4} \frac{EI}{\rho A} \rightarrow \omega = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega = \frac{\pi^2}{(1500)^2} \sqrt{\frac{44 \times 10^{12}}{28300}} = 0.17 \text{ rads/sec}$$

$$f = \frac{\omega}{2\pi} = 0.027 \text{ Hz} \qquad T = \frac{1}{f} = 36 \text{ seconds.}$$

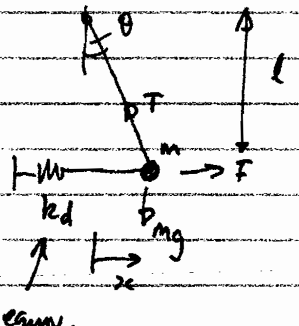
Now include deck stiffening by pendulum effects.



$l = 140\text{m}.$

$$\omega_{pend} = \sqrt{g/l} = \sqrt{\frac{9.81}{140}} = 0.26 \text{ rads/sec.}$$

Combine deck + cables



Stiffness of deck k_d

$$T \approx mg$$

$$m\ddot{x} = F = -mg\sin\theta - k_d x$$

$$\rightarrow \quad \theta \simeq x/l$$

$$\ddot{x} + \underbrace{\left(\frac{g}{l} + \frac{k_d}{m}\right)}_{\omega^2} x = 0$$

and $\omega^2 = \omega_{pend}^2 + \omega_{\substack{deck\\bend}}^2$

$$\omega \simeq \sqrt{0.26^2 + (0.17)^2} = \underline{0.31 \text{ rads/sec}}$$

(with a sizeable proportion of stiffness coming from pendulum effects).

$\therefore$ Nat freq $= \frac{\omega}{2\pi} = \underline{\underline{0.049 \text{ Hz}}}$, Nat period $= \underline{\underline{20.2 \text{ seconds}}}$.

b)

Force/unit length $= \frac{1}{2}\rho V^2 C_D h = \frac{1}{2}(1.25)(50)^2(0.59)(2.9)$

$= \underline{2673 \text{ N/m}}$

Central deflection $= \dfrac{5wL^4}{384 EI}$

If use $(EI)_{basic} \rightarrow \dfrac{5(2673)(1500)^4}{384(44 \times 10^{12})} = \underline{\underline{4 \text{ metres}}}$

but actually stiffer due to pendulum effects, in proportions

0.26^2	of	0.17^2
0.0676		0.0289
70%		30%
pendulum		deck bending.

so bridge is $\frac{10}{3}$ times stiffer due to pendulum effects.

$\therefore$ Central deflection $= \frac{3}{10} \times 4$ metres $=$ $\underline{\underline{1.2\text{ m}}}$

c). $\sigma_x = 0.49$ m.

$\therefore$ Design for $X_{design} = X_{mean} + g\,\sigma_x$

3.5 say for gust factor. (ASSUMPTION)

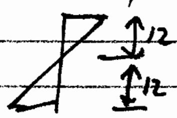
$$= 1.2\text{m} + (3.5)(0.49)$$
$$= \underline{\underline{2.91\text{ m}}}$$

$\therefore$ Mode shape $= 2.91 \sin\left(\frac{\pi x}{L}\right)$

Curvature at centre $= \frac{\pi^2}{L^2}(2.91)$

$=$ slope of strain diagram

$\therefore$ Extreme fibre stress $= \underbrace{12\text{m}} \times K \times E = 12 \times \frac{\pi^2}{(1500)^2}(2.91)(210\times10^3)$ MPa



$= \underline{\underline{32\text{ MPa}}}$