2007 IIB 4F3 Prof J Maciejowski
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Module 4F3: Nonlinear and Predictive Control
Solutions 2007

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1. (a) A subset S of the state space is an *invariant set* if

$$x(t) \in S \Rightarrow x(t+\tau) \in S \quad \text{for all} \quad \tau > 0. \tag{1}$$

Particular kinds of invariant set are (only two asked for):

- Equilibria
- Limit cycles
- Chaotic attractors

(b) Since $0 < \alpha \leq \beta$, both $-1/\alpha$ and $-1/\beta$ are negative. In this case the *Circle criterion* states that if the Nyquist locus of $G(s)$ (evaluated for both positive and negative frequencies) does not penetrate the interior of the circle whose centre is $-\frac{1}{2}(1/\alpha + 1/\beta)$ and for which the segment of the real line $-1/\alpha \leq z \leq -1/\beta$ is a diameter, and if the Nyquist locus encircles the circle as many times counterclockwise as there are unstable poles in $G(s)$, then the feedback system shown in Fig. 1 is globally asymptotically stable.

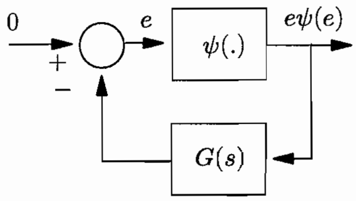

Figure 1:

(c) Chasing the block-diagram of Fig. 2 through, with $e\psi(e)$ as input and e as output, gives

$$G(s) = (k + H(s))\frac{1}{s} \tag{2}$$

since transfer functions in parallel can be combined by adding.
For stability analysis of the feedback loop, only the (nonlinear equivalent of) the 'return ratio' matters, and this remains the same when Fig.2 is replaced by Fig.1 with this $G(s)$. (That is, the fact that the output variable has changed does not matter.)

(d) Since $\text{Im}[G_0(j\omega)] < 0$ for $\omega > 0$, the Nyquist locus of $G_0(s)$ cannot encircle the circle. But we are told that the conditions of the circle criterion are satisfied, hence $G(s)$ must be stable. But the poles

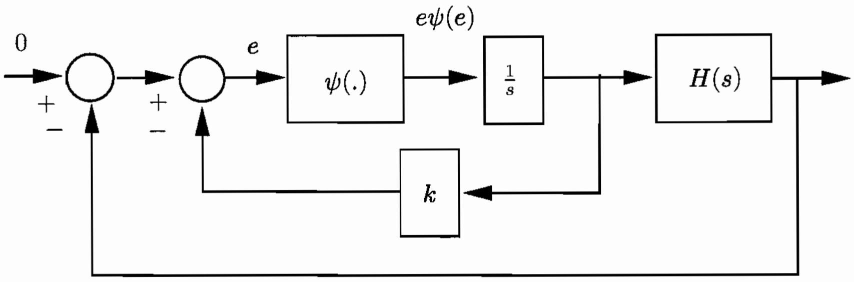

Figure 2:

of $G(s)$ are the poles of $H(s)$, plus the pole at the origin due to the $1/s$ factor. Thus $H(s)$ must be stable.

Let $G_0(s) = (k_0 + H(s))/s$ and $G(s) = (k + H(s))/s$. Then

$$G(s) = G_0(s) + \frac{k - k_0}{s} \tag{3}$$

But $k > k_0$, so $\text{Im}[G(j\omega) - G_0(j\omega)] < 0$ for $\omega > 0$, and $\text{Re}[G(j\omega) - G_0(j\omega)] = 0$ since $k - k_0$ is real. Since $\text{Im}[G_0(j\omega)] < 0$ for $\omega > 0$ and $G_0(j\omega)$ does not penetrate the circle (by assumption), the Nyquist locus of $G(j\omega)$ will be further away from the circle than the locus of $G_0(j\omega)$, and thus will also not penetrate the circle. Furthermore, neither locus can encircle the circle, so the number of encirclements of the circle remains the same. Therefore $G(j\omega)$ satisfies the conditions of the circle criterion(since $G_0(j\omega)$ does).

A figure like Fig. 3 would go a long way to answering this part of the question.

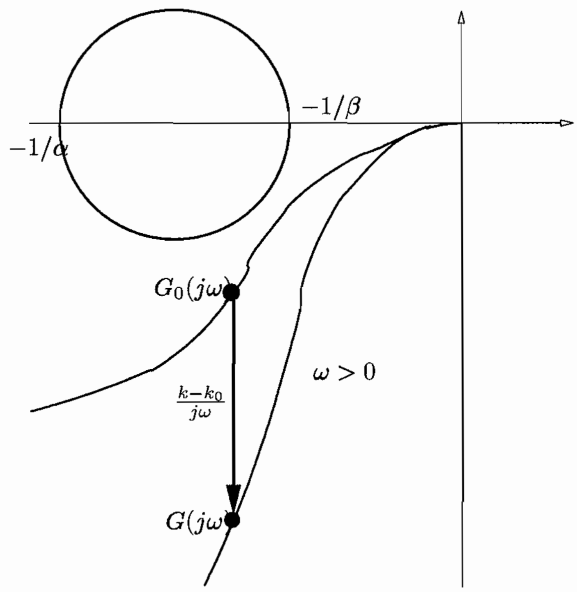

Figure 3: For question 1(d)

2. (a) *LaSalle's Theorem:* Let $S \subseteq \Re^n$ be a compact invariant set. Assume there exists a differentiable function $V : S \to \Re$ such that

$$\dot{V}(x) \leq 0 \; \forall x \in S$$

Let M be the largest invariant set contained in $\{x \in S \mid \dot{V}(x) = 0\}$. Then all trajectories starting in S approach M as $t \to \infty$.

That is the formal statement of LaSalle's Theorem as given in lectures. A less formal statement which captures the main idea, or a statement of the commonly used corollary (for equilibria rather than general invariant sets) is also acceptable. Note that LaSalle's Theorem as stated here does not require $V(x) > 0$ *for* $x \neq 0$.

LaSalle's Theorem is often more useful than Lyapunov's asymptotic stability theorem because the latter relies on establishing that $\dot{V}(x) < 0$ for all $x \neq 0$ for some candidate Lyapunov function V. LaSalle's theorem allows $\dot{V}(x) = 0$ to occur for some $x \neq 0$, providing that it can be proved that this happens only for a finite time interval, after which $\dot{V}(x) < 0$ again. This frequently occurs for engineering systems, with simple energy-related Lyapunov functions.

(b) Let $x_1 = \psi$, $x_2 = \dot{\psi}$. Then the differential equation is equivalent to the state-space equations:

$$\dot{x}_1 = x_2 \tag{4}$$
$$\dot{x}_2 = -ax_2|x_2| + u \tag{5}$$

(c) Consider the proposed function

$$V(\psi, \dot{\psi}) = V(x_1, x_2) = kx_1^2 + x_2^2 \tag{6}$$

Clearly $V(0,0) = 0$ and $V(x_1, x_2) > 0$ for $[x_1, x_2] \neq 0$ if $k > 0$.

With the proportional feedback we have $u = -kx_1$, hence at an equilibrium we have $\dot{x}_1 = 0 \Rightarrow x_2 = 0$ and $\dot{x}_2 = 0 \Rightarrow ax_2|x_2| - kx_1 = 0 \Rightarrow x_1 = 0$, since $k > 0$. Thus $(0,0)$ is the only equilibrium for the closed loop. Now

$$\dot{V} = 2kx_1\dot{x}_1 + 2x_2\dot{x}_2 \tag{7}$$
$$= 2kx_1x_2 + 2x_2(-ax_2|x_2| - kx_1) \tag{8}$$
$$= -2ax_2^2|x_2| \leq 0 \quad \text{since} \quad a > 0. \tag{9}$$

This establishes that V is a Lyapunov function, and hence that $x = 0$ is a stable equilibrium. It does not establish asymptotic stability, because we have $\dot{V} = 0$ whenever $x_2 = 0$, even if $x_1 \neq 0$.

Suppose that $x_1(t_0) \neq 0, x_2(t_0) = 0$, so that $\dot{V}(x(t_0)) = 0$. Then $\dot{x}_2(t_0) = -kx_1(t_0) \neq 0$. Hence there exists some $\tau > 0$ such that $x_2(t_0 + \tau) \neq 0$, and hence $\dot{V}(x(t_0 + \tau)) < 0$. Thus the possibility of a limit cycle is excluded, so 0 is the only invariant set on which $\dot{V} = 0$, and hence by LaSalle's Theorem asymptotic stability of $x = 0$ is proved.

(d) Now we have $u = -\text{sat}(kx_1)$. Note that for $|kx_1| < 1$ we have the same feedback as in part 2c, so that asymptotic stability of $x = 0$ is not destroyed by imposing the saturation (since it is a 'local' property). Also it is easy to check that no new equilibrium is created by the saturation. So the only thing that needs to be established is that for any initial condition $x(0)$, the state will eventually get close enough to 0 for the saturation to cease operating.

Consider the state trajectories. Whenever $x_2 > 0$ we have $\dot{x}_1 > 0$, so trajectories move to the right in the top half of the state space, and to the left in the bottom half. Also $x_2 = 0 \Rightarrow \dot{x}_1 = 0$, so the trajectories cross the x_1 axis vertically — downwards when $x_1 > 0$ and upwards when $x_1 < 0$, since $\dot{x}_2 = -kx_1$ when $x_2 = 0$.

The trajectories are horizontal when $\dot{x}_2 = 0$, namely when $ax_2|x_2| = -\text{sat}(kx_1)$. For $kx_1 > 1$ this means whenever $ax_2|x_2| = -1$, which can happen only for $x_2 < 0$, hence for $x_2 = -\sqrt{1/a}$. Similarly for $kx_1 < -1$ the trajectories are horizontal for $x_2 = +\sqrt{1/a}$. When $|kx_1| < 1$ we have $\dot{x}_2 = 0$ when

$ax_2|x_2| = -kx_1$, namely $ax_2^2 = -kx_1$ when $x_1 < 0, x_2 > 0$, and $ax_2^2 = +kx_1$ when $x_1 > 0, x_2 < 0$. This gives two segments of parabolae — Figure 4 shows the locus of points at which $\dot{x}_2 = 0$ (dashed line). It is easy to see that the trajectories move downwards ($\dot{x}_2 < 0$) at points above this locus, and upwards ($\dot{x}_2 > 0$) below this locus.

Finally, consider the state trajectories far away from 0. For $|x_2| \gg \sqrt{1/a}$ we have $\dot{x}_2 \approx -ax_2|x_2|$, hence $|\dot{x}_2| \gg |\dot{x}_1|$, so for large $|x_2|$ the trajectories become nearly vertical (downwards for $x_2 > 0$, upwards for $x_2 < 0$).

Putting all these considerations together gives typical state trajectories as shown in Figure 4.

Note: It is not expected that candidates will write so much text in answering this part. The expected answer will be mostly the sketch, with some brief annotations.

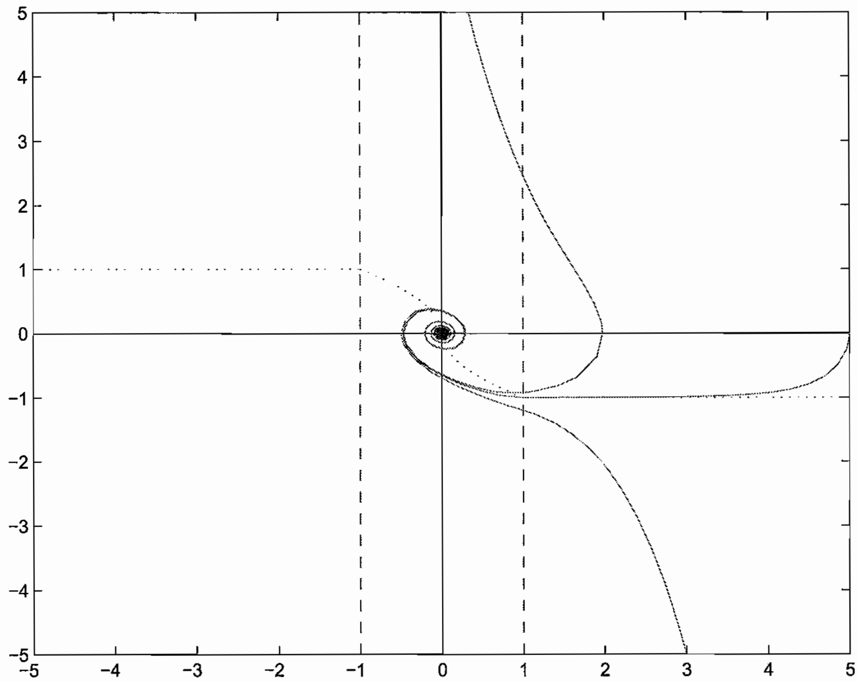

Figure 4: State trajectories for Question 2(d) — with $a = 1$, $k = 1$. The vertical dashed lines show $kx_1 = \pm 1$. The dotted line is the locus of points at which trajectories are horizontal ($\dot{x}_2 = 0$).

3. (a)

$$V(f[x, u_0^*(x)], 0) = \ell(f[x, u_0^*(x)], 0) + F(f[f[x, u_0^*(x)], 0])$$

So the result to be shown is correct if and only if

$$V^*(x) = \ell(x, u_0^*(x)) + F(f[x, u_0^*(x)]).$$

Now

$$V^*(x) = V(x, u_0^*(x)) = \ell(x, u_0^*(x)) + F(f[x, u_0^*(x)]), \quad (10)$$

which shows that the result in the question is correct.

(b) Since $V^*(x) \leq V(x, 0)$ for all x, we have

$$\begin{aligned} V^*(f[x, u_0^*(x)]) - V^*(x) &\leq V(f[x, u_0^*(x)], 0) - V^*(x) && (11) \\ &= -\ell(x, u_0^*(x)) - F(f[x, u_0^*(x)]) + \\ &\quad +\ell(f[x, u_0^*(x)], 0) + F(f[f[x, u_0^*(x)], 0]) && (12) \\ &\leq -\ell(x, u_0^*(x)) && (13) \end{aligned}$$

where the result from part (a) has been used in the second line, and the final line follows from the assumption that

$$F(f[x, 0]) - F(x) + \ell(x, 0) \leq 0 \quad (14)$$

for all x.

(c) To be a Lyapunov function, $V^*(\cdot)$ should be nonincreasing along trajectories of the closed-loop system. This is ensured by the condition that

$$\ell(x, u) \geq 0 \text{ for all } x, u. \quad (15)$$

In addition, in some neighbourhood of $(0, 0)$, the conditions $V^*(x) > 0$ for all $x \neq 0$, and $V(0) = 0$, are needed.

(d) The uncontrolled system has the input fixed at $u = 0$. Thus we can write the uncontrolled behaviour as

$$x(k + 1) = f[x(k), 0] = f_o[x(k)]. \quad (16)$$

(Perhaps I'm being too pedantic here.)

The uncontrolled system has an equilibrium at $x = 0$ since $0 = f[0, 0] = f_o[0]$. We are told that $F(0) = 0$ and $F(x) > 0$ for all $x \neq 0$. So $F(\cdot)$ is a Lyapunov function for the uncontrolled system if $F(f_o[x]) - F(x) \leq 0$ for all x. But from (14) and the condition (added in (15)) that $\ell(x, u) \geq 0$, we have

$$F(f_o[x]) - F(x) \leq -\ell(x, 0) \leq 0. \quad (17)$$

Thus $F(\cdot)$ *is* a Lyapunov function for the uncontrolled system, which is thus proved to have a stable equilibrium at $x = 0$.

4. (a)

$$x_{s+1} = Ax_s + B\{Kx_s + v_s\} \tag{18}$$
$$= (A+BK)x_s + B(k)v_s \tag{19}$$

Thus we get

$$x_1 = (A+BK)x_0 + Bv_0 \tag{20}$$
$$x_2 = (A+BK)x_1 + Bv_1 \tag{21}$$
$$= (A+BK)[(A+BK)x_0 + Bv_0] + Bv_1 \tag{22}$$
$$= (A+BK)^2 x_0 + (A+BK)Bv_0 + Bv_1 \tag{23}$$

Thus we have $X = \Phi x_0 + \Gamma V$ if:

$$\Phi = \begin{bmatrix} A+BK \\ (A+BK)^2 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} B & 0 \\ (A+BK)B & B \end{bmatrix} \tag{24}$$

(b) The given constraints are

$$Cu_0 \le e, \quad Cu_1 \le e, \quad Dx_2 \le f. \tag{25}$$

The first two constraints can be written as

$$C(Kx_0 + v_0) \le e, \quad C(Kx_1 + v_1) \le e. \tag{26}$$

All three constraints can therefore be written in the form

$$Ex_0 + FX + GV \le g \tag{27}$$

if E, F, G, g each has 3 rows (one for each constraint), and are defined as follows:

$$E = \begin{bmatrix} CK \\ 0 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ CK & 0 \\ 0 & D \end{bmatrix}, \quad G = \begin{bmatrix} C & 0 \\ 0 & C \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} e \\ e \\ f \end{bmatrix}. \tag{28}$$

Note: The solution to this part is not unique, but the idea is to translate the constraints into the form of (27) as simply as possible.

(c) Now we need to eliminate the 'FX' term from (27). From part (a) we have $X = \Phi x_0 + \Gamma V$. Substituting into (27) gives

$$Ex_0 + F(\Phi x_0 + \Gamma V) + GV \le g \tag{29}$$

or

$$(F\Gamma + G)V \le g - (E + F\Phi)x_0 \tag{30}$$

hence we have

$$S = F\Gamma + G, \quad T = -(E + F\Phi), \quad h = g \tag{31}$$

(d) A quadratic program (QP) is an optimisation problem of the form

$$\min_\theta \theta^T H\theta + d^T\theta, \text{ subject to } A\theta \le b \tag{32}$$

where A and H are matrices, and b, d are vectors.
The X^TQX term in the cost function can be rewritten, using $X = \Phi x_0 + \Gamma V$, as

$$(\Phi x_0 + \Gamma V)^T Q(\Phi x_0 + \Gamma V) = V^T\Gamma^T Q\Gamma V + 2x_0^T\Phi^T Q\Gamma V + x_0^T\Phi^T Q\Phi x_0 \tag{33}$$

and the final term in this expression can be ignored when minimising, since it does not depend on V.

So to put the minimisation of $J(x(k), V)$, subject to constraints $SV \leq h + Tx(k)$, into the form of a QP requires the conversion of the term $|c^T V|$ into a linear term. For this purpose introduce a new scalar variable e, replace the cost $J(x(k), V)$ by the cost $X^T Q X + e$, with the new (additional) constraints $c^T V \leq e$, $-c^T V \leq e$, and $e \geq 0$, and minimise the cost with respect to both V and e.

Note: This procedure for converting the $|c^T V|$ term into a linear one is essentially the same as that introduced in lectures for converting an absolute-value cost into a linear program. Notice that the condition $e \geq 0$ is essential for this to work — most candidates missed this out in the exam.