Q1

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Bookwork. and lecture notes.

Q2

1^{st} part is bookwork 20%.

Using Bayes' theorem,

$$p(\lambda|n) = \frac{p(\lambda)\, p(n|\lambda)}{\int_0^\infty p(\lambda)\, p(n|\lambda)\, d\lambda}.$$

The combination of the prior + likelihood gives

$$p(\lambda)\, p(n|\lambda) = e^{-\lambda} \frac{\lambda^n}{n!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}.$$

The denominator is

$$\int_0^\infty \frac{\beta^\alpha}{n!\,\Gamma(\alpha)} \lambda^{n+\alpha-1} e^{-(\beta+1)\lambda}\, d\lambda$$

$$= \frac{\beta^\alpha}{n!\,\Gamma(\alpha)} \int_0^\infty \lambda^{n+\alpha-1} e^{-(\beta+1)\lambda}\, d\lambda.$$

We therefore need an integral of the form

$$\int_0^\infty x^p e^{-\gamma x}\, dx$$

Q2 cont

$$\int_0^\infty e^{-\gamma x} dx = \frac{1}{\gamma}$$

$$\frac{\partial^p}{\partial \gamma^p} \left(\int_0^\infty e^{-\gamma x} dx \right) = \int_0^\infty x^p e^{-\gamma x} dx.$$

$$\frac{\partial^p}{\partial \gamma^p} \left(\frac{1}{\gamma} \right) = \frac{p!}{\gamma^{p+1}}$$

Therefore $p(\lambda|n) = \frac{(\beta+1)^{\alpha+n}}{\Gamma(\alpha+n)} \lambda^{\alpha+n-1} e^{-(\beta+1)\lambda}$.

which is another gamma distribution with parameters $\alpha+n$, $\beta+1$. and is therefore a conjugate distribution 40%.

b) $\hat{\lambda}_{mmse}(n) = \int_0^\infty \lambda \, p(\lambda|n) d\lambda = \frac{\alpha+n}{\beta+1}$ 20%.

c) in limit $\alpha, \beta \to 0$

$\hat{\lambda} = n$ the "classical" result 20%.

Q 3

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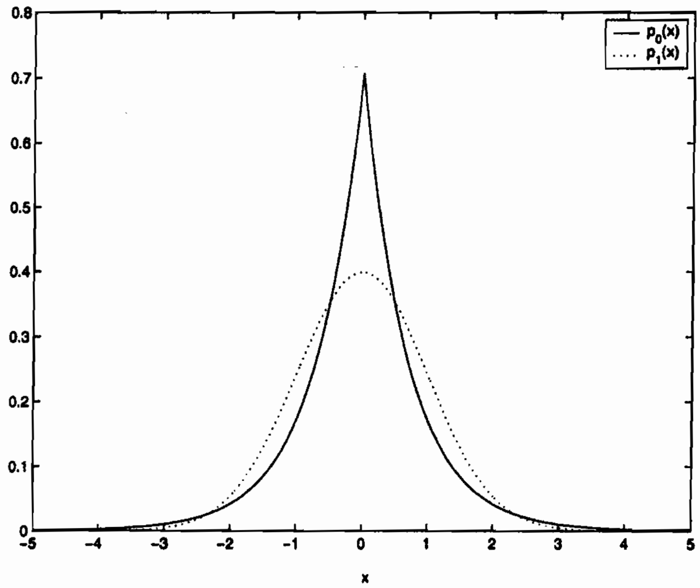
a. The means are zero for both densities because $xf(x)$ is an odd function. The variance H_1 is 1 because we identify the density as being that of a normal random variable w mean zero and variance one. Under H_0, the variance is

$$\begin{aligned}
\mathrm{Var}[X|H_0] &= \int_{-\infty}^{0} \frac{x^2}{\sqrt{2}} e^{\sqrt{2}x} dx + \int_0^{\infty} \frac{x^2}{\sqrt{2}} e^{-\sqrt{2}x} dx = 2\int_0^{\infty} \frac{x^2}{\sqrt{2}} e^{-\sqrt{2}x} dx \\
&= 2\left\{ \frac{-x^2}{2} e^{-\sqrt{2}x} - \frac{x}{\sqrt{2}} e^{-\sqrt{2}x} - \frac{e^{-\sqrt{2}x}}{2} \right\}_0^{\infty} \text{ (integration by parts)} \\
&= 2 \times \frac{1}{2} = 1.
\end{aligned}$$



b. Likelihood ratio:

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|}} = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2 + \sqrt{2}|x|}$$

Decision regions:

$$\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2 + \sqrt{2}|x|} \begin{cases} > \eta, & \text{choose } H_1 \\ < \eta, & \text{choose } H_0 \end{cases}$$

Q3 cont.

Taking logarithm of both sides,

$$-\frac{x^2}{2}+\sqrt{2}|x|\begin{cases} >\ln(\sqrt{\pi}\eta), & \text{choose } H_1\\ <\ln(\sqrt{\pi}\eta), & \text{choose } H_0\end{cases}$$

Since $x^2=|x|^2$, we have

$$|x|^2-2\sqrt{2}|x|+2\ln(\sqrt{\pi}\eta)\begin{cases} >0, & \text{choose } H_0\\ <0, & \text{choose } H_1\end{cases}.$$

The left hand side is a quadratic polynomial in $|x|$, so we can solve for $|x|$ using the quadratic formula. If the discriminant of the LHS is negative, the LHS is always positive and therefore we always choose H_0. In other words, if

$$(2\sqrt{2})^2-4(2)(\ln(\sqrt{\pi}\eta))<0 \iff 1<\ln(\sqrt{\pi}\eta) \iff \eta>\frac{e}{\sqrt{\pi}},$$

we always choose H_0.
If $0<\eta<\frac{e}{\sqrt{\pi}}$, the discriminant is positive. Solving for $|x|$,

$$|x|=\sqrt{2}\pm\sqrt{2}\sqrt{1-\ln(\sqrt{\pi}\eta)}$$

Two cases arise depending on whether $1-\ln(\sqrt{\pi}\eta)$ is greater or lesser than 1.
If

$$1-\ln(\sqrt{\pi}\eta)>1 \iff \eta<\frac{1}{\sqrt{\pi}},$$

then the decision regions become

$$H_0 \leftarrow \;\Big|_{-\sqrt{2}-\sqrt{2-2\ln(\sqrt{\pi}\eta)}}\; H_1 \;\Big|_{\sqrt{2}+\sqrt{2-2\ln(\sqrt{\pi}\eta)}}\; H_0 \rightarrow \quad \rightarrow x$$

If

$$0<1-\ln(\sqrt{\pi}\eta)<1 \iff \frac{1}{\sqrt{\pi}}<\eta<\frac{e}{\sqrt{\pi}},$$

the decision regions are

$$H_0 \leftarrow \Big|_{-a-b}\; H_1 \;\Big|_{-a+b}\; H_0 \;\Big|_{a-b}\; H_1 \;\Big|_{a+b}\; H_0 \rightarrow \quad \rightarrow x$$

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where $a=\sqrt{2}$ and $b=\sqrt{2-2\ln(\sqrt{\pi}\eta)}$. So the critical values at which the nature of the decision regions changes are $\eta_1=1/\sqrt{\pi}$ and $\eta_2=e/\sqrt{\pi}$.

Q4

Suppose that we want to decide whether or not a coin is fair by tossing it eight times and observing the number of heads showing up. Assume that we have to decide in favor of one of the following two hypotheses:

H_0: Fair coin, $P(\text{head}) = p_0 = \frac{1}{2}$

H_1: Unfair coin, $P(\text{head}) = p_1 = 0.4$

a. Derive the MAP devision rule assuming $P(H_0) = \frac{1}{2}$.

b. Calculate the average probability of error.

Solution:

The observed random variable Y is the number of "heads" showing up by tossing the coin eight times.

- Pdf of Y under both hypothesis:

$$\begin{aligned} H_0: \quad P[Y = n|H_0] &= \binom{8}{n} p_0^n (1-p_0)^{8-n} \quad \text{with} \quad p_0 = 0.5 \\ H_1: \quad P[Y = n|H_1] &= \binom{8}{n} p_1^n (1-p_1)^{8-n} \quad \text{with} \quad p_1 = 0.4 \end{aligned}$$

- Likelihood ratio:

$$\begin{aligned} L(n) &= \frac{P[Y = n|H_1]}{P[Y = n|H_0]} \\ &= \frac{p_1^n (1-p_1)^{8-n}}{p_0^n (1-p_0)^{8-n}} \\ &= \left(\frac{p_1}{p_0}\right)^n \left(\frac{1-p_0}{1-p_1}\right)^n \left(\frac{1-p_1}{1-p_0}\right)^8 \\ L(n) &= \left(\frac{p_1}{p_0} \cdot \frac{1-p_0}{1-p_1}\right)^n \left(\frac{1-p_1}{1-p_0}\right)^8 \end{aligned}$$

- Log-likelihood ratio:

$$\begin{aligned} l(n) &= \ln[L(n)] \\ &= n \cdot [\ln \frac{1-p_0}{p_0} - \ln \frac{1-p_1}{p_1}] + 8 \ln\left(\frac{1-p_1}{1-p_0}\right) \end{aligned}$$

Q4 cont.

- MAP decision rule:

$$\begin{aligned} P[H_0] &= P[H_1] = \frac{1}{2} \\ \Rightarrow \gamma &= \frac{P[H_0]}{P[H_1]} = 1 \end{aligned}$$

$$\begin{aligned} l(n) &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\gamma \\ n\cdot\left[\underbrace{\ln\frac{1-p_0}{p_0}}_{=0} - \ln\frac{1-p_1}{p_1}\right] + 8\ln\left(\frac{1-p_1}{1-p_0}\right) &\underset{H_0}{\overset{H_1}{\gtrless}} 0 \\ n\cdot\ln\left(\frac{1-p_1}{p_1}\right) &\underset{H_0}{\overset{H_1}{\lessgtr}} 8\cdot\ln\left(\frac{1-p_1}{1-p_0}\right) \\ n &\underset{H_0}{\overset{H_1}{\lessgtr}} \underbrace{8\cdot\frac{\ln\left(\frac{1-p_1}{1-p_0}\right)}{\ln\left(\frac{1-p_1}{p_1}\right)}}_{=\tilde{\gamma}} \end{aligned}$$

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Evaluation of the threshold $\tilde{\gamma}$:

$$\begin{aligned} \tilde{\gamma} &= 8\cdot\frac{\ln\left(\frac{1-p_1}{1-p_0}\right)}{\ln\left(\frac{1-p_1}{p_1}\right)} \\ &= 8\cdot\frac{\ln\left(\frac{0.6}{0.5}\right)}{\ln\left(\frac{0.6}{0.4}\right)} \\ &= 8\cdot\frac{\ln 1.2}{\ln 1.5} \\ &\approx 3.6 \end{aligned}$$

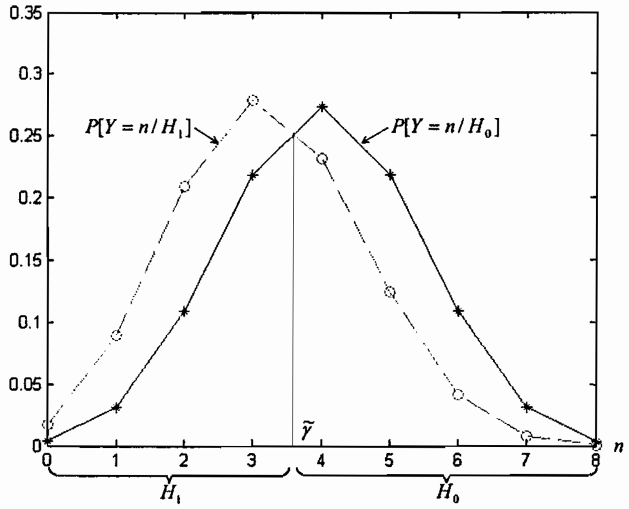
$$n \underset{H_0}{\overset{H_1}{\lessgtr}} 8\cdot\frac{\ln 1.2}{\ln 1.5}$$

- Probability of incorrect decision P_e:

$$\begin{aligned} P_e &= P[D=H_1|H_0]\cdot P[H_0] + P[D=H_0|H_1]\cdot P[H_1] \\ &= 0.5\cdot\{P[D=H_1|H_0] + P[D=H_0|H_1]\} \end{aligned}$$

$$\begin{aligned} P[D=H_1|H_0] &= P[Y<3.6|H_0] \\ &= \sum_{n=0}^{3}\binom{8}{n}p_0^n(1-p_0)^{8-n} \\ &\approx 0.3633 \end{aligned}$$

Q4 cont.



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\begin{aligned}
P[D = H_0|H_1] &= P[Y > 3.6|H_1] \\
&= \sum_{n=4}^{8} \binom{8}{n} p_1^n (1-p_1)^{8-n} \\
&\approx 0.4059
\end{aligned}
$$

$$
\begin{aligned}
P_e &\approx 0.5 \cdot (0.3633 + 0.4059) \\
P_e &\approx 0.3846
\end{aligned}
$$

50%