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1 (a) Let $I = \int_a^b F(y, y', x)\,dx$

Perturb $y(x) \to y(x) + \delta y(x)$
Substitute and ignore terms $O(\delta y^2)$ or higher, to obtain an expression of the form

$$\delta I \simeq \int_a^b \left(A(x)\,\delta y + B(x)\,\delta y'\right) dx$$

Integrate by parts on second term $\to \left[B(x)\,\delta y\right]_a^b - \int_a^b B'\delta y\,dx$

Then argue that if $\int_a^b (\sim\!\sim)\,\delta y\,dx = 0$

for any perturbation δy, then $(\sim) = 0$
Also the boundary term must vanish, so at $x=a$ and $x=b$, either $\delta y = 0$ (i.e. 'fixed boundary condition') or $B = 0$ ("free" boundary) [20%]

(b) Total kinetic energy is T_{beam}, so
$$\tilde{T} = \frac{m}{2}\int_0^L u^2\,dx$$

Total potential energy $V = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{1}{2}K\left(\frac{du}{dx}(0)\right)^2 + \frac{1}{2}K\left(\frac{dy}{dx}(L)\right)^2$

So Rayleigh quotient $R = \frac{V}{\tilde{T}} = \frac{EI\int u''^2\,dx + Ku'^2(0) + Ku'^2(L)}{m\int u^2\,dx}$

Let $u \to u + \delta u$, substitute and ignore $O(\delta u^2)$:

$$R+\delta R = \frac{V+\delta V}{\tilde{T}+\delta\tilde{T}} \simeq (V+\delta V)\cdot\frac{1}{\tilde{T}}\left(1+\frac{\delta\tilde{T}}{\tilde{T}}\right)^{-1} \simeq \frac{1}{\tilde{T}}(V+\delta V)\left(1-\frac{\delta\tilde{T}}{\tilde{T}}\right)$$

$$\simeq \frac{V}{\tilde{T}} + \frac{\delta V}{\tilde{T}} - \frac{V}{\tilde{T}^2}\delta\tilde{T}$$

where $\delta V \simeq EI\int u''\,\delta u''\,dx + Ku'(0)\,\delta u'(0) + Ku'(L)\,\delta u'(L)$

and $\delta\tilde{T} \simeq m\int u\,\delta u\,dx$, and $\frac{V}{\tilde{T}} = \omega_n^2$

$$\therefore\ \delta R \simeq \frac{1}{T}\left[EI\int u''\delta u''dx + Ku'(0)\delta u'(0) + Ku'(L)\delta u'(L) - m\omega_n^2\int u\,\delta u\,dx\right]$$

But $$\int_0^L u''\delta u''dx = \left[u''\delta u'\right]_0^L - \int_0^L u'''\delta u' dx$$

$$= \left[u''\delta u'\right]_0^L - \left[u'''\delta u\right]_0^L + \int_0^L u''''\delta u\,dx$$

So if $\delta R = 0$ for all possible $\delta u(x)$, use argument of calculus of variations from part (a). From the two integral terms, deduce that $EI u'''' - m\omega_n^2 u = 0$.

Boundary terms require:

$$EI\left[u''\delta u'\right]_0^L - EI\left[u'''\delta u\right]_0^L + Ku'(0)\delta u'(0) + Ku'(L)\delta u'(L) = 0$$

Since ends have $u=0$, must have $\delta u = 0$ at $0, L$.

But u' is not constrained at the ends, and could take independent values at the two ends. So require:

$$\begin{cases} EIu''(L) + Ku'(L) = 0 & \text{from } \delta u'(L) \text{ terms} \\ -EIu''(0) + Ku'(0) = 0 & \text{from } \delta u'(0) \text{ terms.} \end{cases}$$ [40%]

(c) For $K=0$ this boundary condition reduces to $u''=0$ at $0, L$.

If $u = \sin\frac{n\pi x}{L}$, $u' = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$, $u'' = -\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}$, $u'''' = \left(\frac{n\pi}{L}\right)^4\sin\frac{n\pi x}{L}$

So all boundary conditions are automatically satisfied, and the differential equation is also satisfied provided

$$EI\left(\frac{n\pi}{L}\right)^4 = m\omega_n^2, \quad \text{ie} \quad \omega_n = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m}}$$ [15%]

(d) If K is small, can use the <u>same</u> function $u(x)$ in the Rayleigh quotient, to leading order, because effects of changes to u are 2nd order.

So if new frequency is Ω_n, $\Omega_n^2 \simeq R + \delta R$ with $R = \omega_n^2$

and $\delta R \simeq \dfrac{Ku'^2(0) + Ku'^2(L)}{m\int u^2 dx}$ evaluated with $u = \sin\frac{n\pi x}{L}$

$$\therefore\ \Omega_n^2 \simeq \omega_n^2 + \frac{2K\left(n\pi/L\right)^2}{mL/2} = \omega_n^2 + \frac{4K}{m}\frac{n^2\pi^2}{L^3}$$ [25%]

2 (a) $\nabla(1/r)|_i = \frac{\partial}{\partial x_i}\left((x_j x_j)^{-1/2}\right) = -\frac{1}{2}(x_k x_k)^{-3/2} \cdot 2x_j \delta_{ij}$

$= -(x_k x_k)^{-3/2} \cdot x_i$

ie $\nabla\left(\frac{1}{r}\right) = -\frac{\underline{x}}{|\underline{x}|^3}$

$\nabla^2\left(\frac{1}{r}\right) = \nabla \cdot (\nabla(1/r)) = -\frac{\partial}{\partial x_i}\left[x_i (x_k x_k)^{-3/2}\right]$

$= -\delta_{ii}\left[x_k x_k\right]^{-3/2} + x_i \frac{3}{2}(x_j x_j)^{-5/2} \cdot 2x_k \delta_{ik}$

$= -3\left[x_k x_k\right]^{-3/2} + 3(x_i x_i)(x_j x_j)^{-5/2}$

$= 0$ provided $(x_k x_k) \neq 0$ when it is undefined.

[35%]

(b) (i) If $\underline{x} = 0$ is not within the volume V, then $\nabla^2\left(\frac{1}{r}\right) = 0$ so $\iiint_V \nabla^2(1/r)\, dV = 0$

[15%]

(ii) For the sphere $|\underline{x}| = a$: use divergence theorem

$I = \iiint_V \nabla^2(1/r)\, dV = \iint_{|\underline{x}|=a} \nabla\left(\frac{1}{r}\right) \cdot \underline{dS} = -\iint_{|\underline{x}|=a} \frac{\underline{x} \cdot \underline{dS}}{|\underline{x}|^3}$

But on the surface $|\underline{x}|^3 = a^3$, $\underline{x} \cdot \underline{dS} = a\, dS$

So $I = -\frac{1}{a^2}\iint_{|\underline{x}|=a} dS = -4\pi$

[15%]

(iii) For an arbitrary volume V including the origin, divide it into the largest sphere centred on the origin, plus the remainder of the volume. Call these V_1, V_2 with $V = V_1 + V_2$

Then $I = \iiint_{V_1+V_2} \nabla^2\left(\frac{1}{r}\right) dV = -4\pi + 0 = -4\pi$

[15%]

(e) So the function $-\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)$ behaves like a three-dimensional version of the delta function: any volume integral of the function either gives the answer 1 or 0 depending on whether the origin is inside or outside the volume.

In 1D, the delta function can conveniently be represented as the derivative of the unit step function. Here the 3D delta function is represented as $\nabla \cdot \underline{f}$ where $\underline{f} = \frac{\underline{x}}{4\pi|\underline{x}|^3}$ i.e. as a derivative of an explicit function.

The function $-\frac{1}{4\pi r}$ is the solution of

$$\nabla^2 \phi = \delta(\underline{x}).$$

To solve $\nabla^2\phi = f(\underline{x})$ where f is a given function, the convolution integral can be used:

$$\phi(\underline{x}) = -\frac{1}{4\pi}\iiint \frac{f(\underline{x}')}{|\underline{x}-\underline{x}'|}\, dV'$$

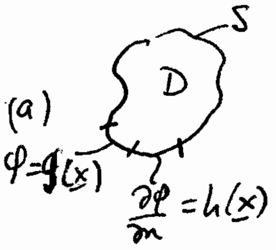
where the element dV' is with respect to $\underline{x}'$

Only valid for Poisson's equation in empty space, because the solution $\frac{-1}{4\pi r} \to 0$ as $r \to \infty$

So that must be the boundary condition for the corresponding Poisson's equation problem.

[20%]

3(a)

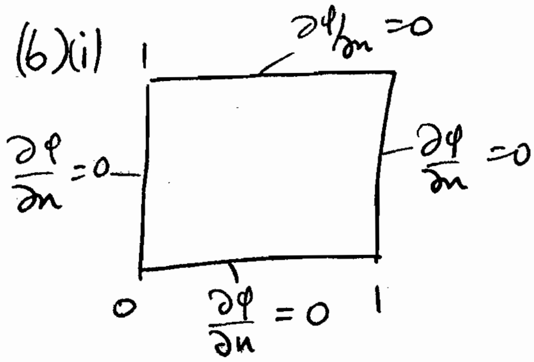


$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \dots = f(\underline{x})$ in D with b.c.'s on S

is well-posed if solution (i) exists (ii) is unique (iii) is stable to "input" changes. i.e. for small changes to f, g, h, etc. the change in u remains small across D.
i.e. bounded

[15%]

(b)(i)



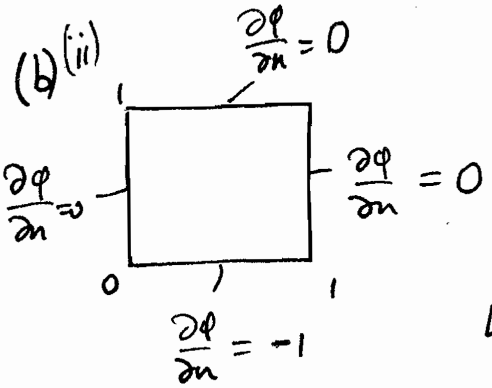
This set of b.c.'s does not determine φ i.e. can add const

Solution not unique $\Rightarrow$ not well posed

Can fix this by specifying some condition to fix the constant e.g. $\varphi(0,0) = 0$
Equation is elliptic & these b.c.'s then suitable.

[15%]

(b)(ii)



$$\int_A \nabla^2 \varphi \, dA = \oint \frac{\partial \varphi}{\partial n} \, dl$$

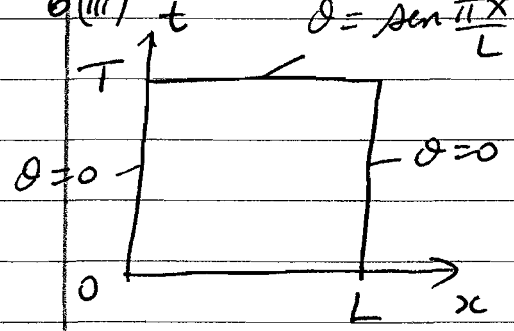
L.H.S. $= \int_0^1 \int_0^1 0 \, dy dx = 0$ R.H.S. $= -1$

$\Rightarrow$ No solution $\Rightarrow$ not well posed

Can make a well posed problem by modifying $\frac{\partial \varphi}{\partial n}$ on boundary s.t. $\oint \frac{\partial \varphi}{\partial n} \, dl = 0$

[30%]

6(iii)



The diffusion equation is parabolic and is integrated forward in time.

This is thus not an appropriate set of b.c.'s, which should have θ specified at $t=0$ (or some other b.c. there).

Solutions to the diffusion equation are of the form

$$e^{-\alpha k^2 t} \sin kx$$

If we take

$$\theta = \underbrace{\sin\frac{\pi x}{L}\, e^{-\frac{\alpha\pi^2}{L^2}(t-T)}}_{\theta_1} + \underbrace{\frac{1}{N^2}\sin\frac{N\pi x}{L}\, e^{-\frac{\alpha N^2\pi^2}{L^2}(t-T)}}_{\delta\theta}$$

then $\frac{\partial^2\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}$, $\theta(0,t) = \theta(L,t) = 0$

and $\theta(x,T) = \sin\frac{\pi x}{L} + \frac{1}{N^2}\sin\frac{N\pi x}{L}$

this represents, for large N, a very small change to the b.c. data but

$$\theta(x,0) = e^{\frac{\alpha\pi^2 T}{L^2}}\sin\frac{\pi x}{L} + \frac{1}{N^2}e^{\frac{\alpha N^2\pi^2 T}{L^2}}\sin\frac{N\pi x}{L}$$

which is arbitrarily large (for large N)

$\therefore$ Solution not stable to small changes in b.c.'s
$\Rightarrow$ not well posed.

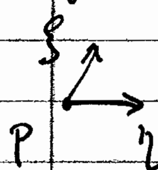
As stated above, can fix this by applying b.c. at $t=0$

[40%]

4(a) For an equation of the form

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

if $b^2 - ac > 0$, the equation is hyperbolic and two independent families of curves $\xi = \text{const}$ and $\eta = \text{const}$ can be found such that at any point P, the equation becomes



$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \dots = 0.$$

The equation can be integrated locally along these curves (characteristics) as a pair of o.d.e.'s. [20%]

(b) $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2y}\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2y^2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2y^3}\frac{\partial u}{\partial y} = 0$$

is of hyperbolic type if $b^2 - ac < 0$

where $a = 1 \quad b = -\frac{1}{4y} \quad c = -\frac{1}{2y^2}$

$$b^2 - ac = \frac{1}{16y^2} + \frac{1}{2y^2} > 0 \text{ for all } y \text{ (except } y = 0)$$

$\therefore$ Equation is hyperbolic [15%]

(c) $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2y}\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2y^2}\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{2y}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial y}\right)u - \frac{1}{2y^3}\frac{\partial u}{\partial y}$$

$\therefore$ Equation (1) is

$$\left(\frac{\partial}{\partial x} + \frac{1}{2y}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial y}\right)u = 0$$

Introduce characteristic variables ξ & η s.t.

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \frac{1}{2y}\frac{\partial}{\partial y}$$

$$\& \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial y}$$

$$\Rightarrow \frac{\partial x}{\partial \xi} = 1 \quad \frac{\partial x}{\partial \eta} = 1 \quad \text{i.e.} \quad \underline{x = \xi + \eta}$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{2y} \quad \frac{\partial y}{\partial \eta} = -\frac{1}{y} \Rightarrow \left.\begin{array}{l} y^2 = \xi + fn(\eta) \\ \frac{y^2}{2} = -\eta + fn(\xi) \end{array}\right\} \Rightarrow \underline{y^2 = \xi - 2\eta}$$

In these variables, equation (1) is

$$\underline{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0} \quad \text{with solution} \quad \underline{u = f(\xi) + g(\eta)}$$ [25%]

(d) $$\left.\begin{array}{l} x = \xi + \eta \\ y^2 = \xi - 2\eta \end{array}\right\} \Rightarrow \xi = \frac{1}{3}(2x + y^2) \quad \eta = \frac{1}{3}(x - y^2)$$

$$\Rightarrow u = f\left(\frac{2x+y^2}{3}\right) + g\left(\frac{x-y^2}{3}\right)$$

$$u(0,y) = 0 \Rightarrow 0 = f\left(\frac{y^2}{3}\right) + g\left(-\frac{y^2}{3}\right)$$

$$\frac{\partial u}{\partial x}(0,y) = y^2 \Rightarrow y^2 = \frac{2}{3} f'\left(\frac{y^2}{3}\right) + \frac{1}{3} g'\left(\frac{y^2}{3}\right)$$

Put $\frac{y^2}{3} = \alpha \Rightarrow 0 = f(\alpha) + g(-\alpha)$

$$3\alpha = \frac{2}{3} f'(\alpha) + \frac{1}{3} g'(-\alpha)$$

Integrating $\frac{3\alpha^2}{2} = \frac{2}{3} f(\alpha) - \frac{1}{3} g(-\alpha) = f(\alpha)$

and $g(\alpha) = -f(-\alpha) = -\frac{3(-\alpha)^2}{2} = -\frac{3\alpha^2}{2}$

Hence
$$u = \frac{3}{2}\left(\frac{2x+y^2}{3}\right)^2 - \frac{3}{2}\left(\frac{x-y^2}{3}\right)^2$$

$$= \frac{3}{2} \cdot \frac{1}{9}\left[4x^2 + 4xy^2 + y^4 - x^2 + 2xy^2 - y^4\right]$$

$$= \frac{1}{6}\left[3x^2 + 6xy^2\right]$$

$$= \underline{xy^2 + \frac{x^2}{2}}$$ [40%]