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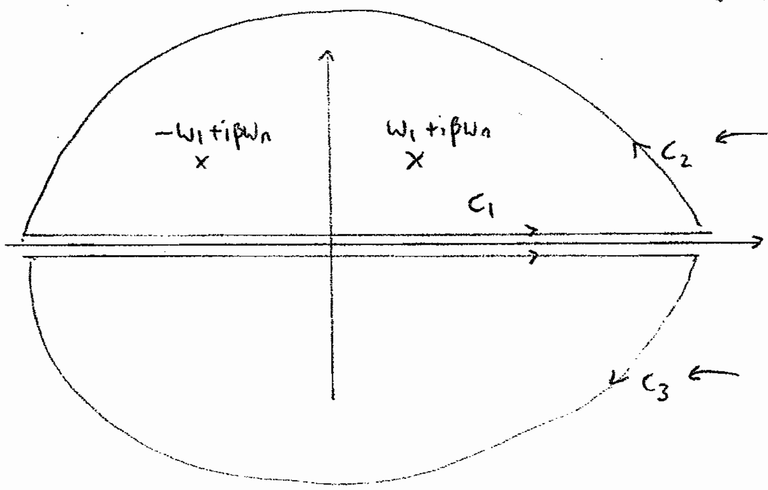
1a) Poles at $\omega_n^2 + 2i\beta\omega_n\omega - \omega^2 = 0 \Rightarrow \omega = \frac{1}{2}\left\{i2\beta\omega_n \pm \sqrt{-4\beta^2\omega_n^2 + 4\omega_n^2}\right\}$

$$\omega = i\beta\omega_n \pm \underbrace{\omega_n\sqrt{1-\beta^2}}_{\text{call this } \omega_1}$$



$e^{i\omega t} \to 0$ here for $t>0$ $\Rightarrow$ use for $t>0$

$e^{i\omega t} \to 0$ here for $t<0$ $\Rightarrow$ use for $t<0$

For $t>0$: $\int_{C_1+C_2} = \int_{C_1} = 2\pi i \times \sum \text{residues} = x(t)$

Integrand is $\dfrac{e^{i\omega t}}{-[\omega_1 + i\beta\omega_n - \omega][\omega_1 - i\beta\omega_n + \omega]} \times \dfrac{1}{2\pi}$

Residue at $\omega = \omega_1 + i\beta\omega_n$ is $\dfrac{e^{i\omega_1 t - \beta\omega_n t}}{-2\omega_1 2\pi}$

Residue at $\omega = -\omega_1 + i\beta\omega_n$ is $\dfrac{e^{-i\omega_1 t - \beta\omega_n t}}{2\omega_1 2\pi}$

$\Big\}\ \sum \text{residues} = -\dfrac{i}{\omega_1}\sin\omega_1 t \, . \, e^{-\beta\omega_n t}\left(\dfrac{1}{2\pi}\right)$

$$\Rightarrow \int_{C_1} + \int_{C_2} = 2\pi i \times \left(\frac{-i}{\omega_1}\right)\sin\omega_1 t \, e^{-\beta\omega_n t} \times \left(\frac{1}{2\pi}\right)$$

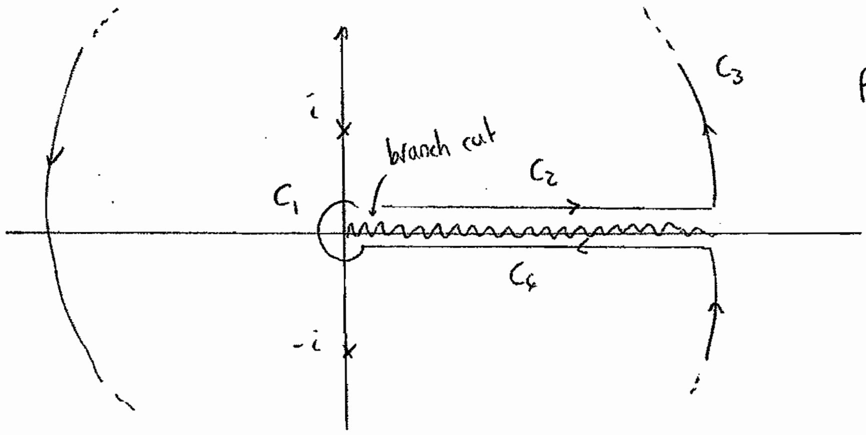
$$\Rightarrow \underline{x(t) = \frac{1}{\omega_1} e^{-\beta\omega_n t}\sin\omega_1 t}$$

For $t<0$: $\int_{C_1} + \int_{C_3} = \int_{C_1} = -2\pi i \times \sum \text{residues} \ \underline{= 0}$

$$\Rightarrow \underline{x(t) = 0}$$

[50%]

b)



Poles at $z = \pm i$

On C_3 : $\int_{C_3} \to \int \frac{\ln(z)}{z^2}\,dz \to 0$

on C_1 : $\int_{C_1} \to \int \frac{\ln(z)}{1}\,dz \to 0$

on C_2 : $\ln(z) = \ln(x)$

on C_4 : $\ln(z) = \ln(x) + 2\pi i$

$$\Rightarrow \quad \int_C = \int_{C_2} \frac{\ln(x)}{1+x^2}\,dx + \int_{C_4} \frac{\ln(x)+2\pi i}{1+x^2}\,dx = \underline{-2\pi i \int_0^\infty \frac{1}{1+x^2}\,dx}$$

Also $\int_C = 2\pi i \times \sum$ residues. Integrand $= \frac{\ln(z)}{(z+i)(z-i)}$ $\xrightarrow{\text{Residue at } i}$ $\frac{\ln(i)}{2i}$

$\xrightarrow{\text{Residue at } -i}$ $\frac{\ln(-i)}{-2i}$

$$\sum \text{residues} = \frac{1}{2i}\left[\underbrace{\ln(i)}_{\frac{\pi}{2}i} - \underbrace{\ln(-i)}_{\frac{3\pi}{2}i}\right]$$

$$\int_C = 2\pi i\left(-\frac{\pi}{2}\right) = -2\pi i \int_0^\infty \frac{1}{1+x^2}\,dx \quad \Rightarrow \quad \underline{\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}}$$

[50%]

2 a) (i) Requirement is that $F'(z) = \lim_{\delta z \to 0} \dfrac{F(z+\delta z) - F(z)}{\delta z}$ is independent of path of δz

$$\left.\begin{array}{ll} \text{Take } \delta z = \delta x \Rightarrow & F'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \\ \text{Take } \delta z = i\delta y \Rightarrow & F'(z) = -i\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} \end{array}\right\} \text{ For equality } \underline{\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{array}}$$

[20%]

(ii) $u = 2xy$

$\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 2y \Rightarrow v = y^2 + g(x)$ $g(x)$: some function of x

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -2x \Rightarrow v = -x^2 + r(y)$ $r(y)$: some function of y

$\Rightarrow v = y^2 - x^2 + \text{const} = y^2 - x^2 + C$ say

$\Rightarrow F = 2xy + i(y^2 - x^2) + Ci = -i(x+iy)^2 + Ci = \underline{-iz^2 + Ci}$

[20%]

b) If $F(z)$ has an n'th order pole at $z = a$, Laurent expansion is:

$$F(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \ldots \frac{a_1}{(z-a)} + a_0 + a_1(z-a) \ldots$$

Extends the Taylor series to singular points. cf, the Taylor Series is :-

$F(z) = a_0 + a_1(z-a) + a_2(z-a)^2$ – divergent if a circle about 'a' encloses a singular point.

[10%]

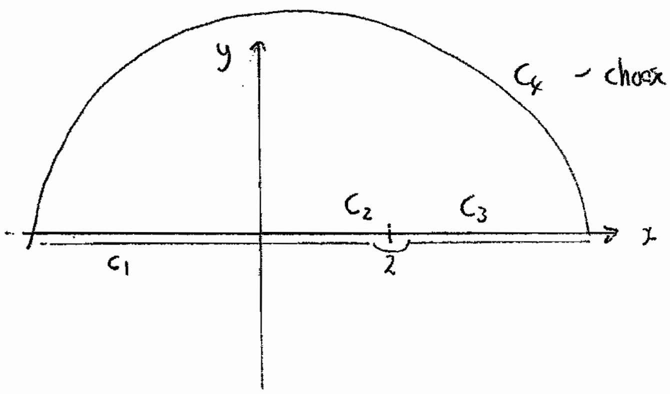
c) (i) $g(z) = g(a) + g'(a)(z-a) + g''(a)(z-a)^2 \tfrac{1}{2} + g'''(a)(z-a)^3 \tfrac{1}{6} + \ldots$

$$\Rightarrow F(z) = \frac{g(a)}{(z-a)^3} + \frac{g'(a)}{(z-a)^2} + \frac{\tfrac{1}{2}g''(a)}{(z-a)} + \tfrac{1}{6}g'''(a) + \ldots$$

Residue = coefficient of $\dfrac{1}{z-a} = \underline{\tfrac{1}{2}g''(a)}$

[20%]

(ii)



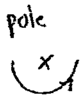
C_4 – chose this because $e^{iz} \to 0$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-2)^3}\, dx = \int_{C_1+C_3} \frac{e^{iz}}{(x-2)^3}\, dz$$

Consider $\displaystyle\int_{C=C_1+C_2+C_3+C_4} \frac{e^{iz}}{(z-2)^3}\, dz = 2\pi i \times \sum \text{residues} = 2\pi i \times (\text{residue at } z=2)$

From part (i), residue at $z=2$ is $\frac{1}{2}\frac{d^2}{dx^2}\left(e^{ix}\right)\Big|_{x=2} = -\frac{1}{2}e^{2i}$

$\Rightarrow \displaystyle\int_C = 2\pi i\left(-\tfrac{1}{2}e^{2i}\right) = \underline{-\pi i e^{2i}}$

Now $\displaystyle\int_{C_4} = 0$ and $\displaystyle\int_{C_2} =$ 

$= \pi i \times \text{residue} = -\frac{1}{2}\pi i e^{2i}$

Thus $\displaystyle\int_{C_1+C_3} = \int_C - \int_{C_4} - \int_{C_2} = -\pi i e^{2i} + \tfrac{1}{2}\pi i e^{2i} = \underline{-\tfrac{1}{2}\pi i e^{2i}}$

$$\Rightarrow \underline{\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x-2)^3}\, dx = -\tfrac{1}{2}\pi i e^{2i}}$$

[30%]

3. (a) $$f(\underline{x}_{k+1}) = f(\underline{x}_k) + \nabla f(\underline{x}_k)^T(\underline{x}_{k+1} - \underline{x}_k) + \frac{1}{2}(\underline{x}_{k+1} - \underline{x}_k)^T \underline{\underline{H}}(\underline{x}_k)(\underline{x}_{k+1} - \underline{x}_k) + R$$

R = higher order terms

Approximate $f(\underline{x}_{k+1})$ by a quadratic, i.e. set $R=0$

Differentiate wrt $\underline{x}_{k+1}$

$$\nabla f(\underline{x}_{k+1}) = \nabla f(\underline{x}_k) + \underline{\underline{H}}(\underline{x}_k)(\underline{x}_{k+1} - \underline{x}_k)$$

If $\underline{x}_{k+1}$ is a minimum $\nabla f(\underline{x}_{k+1}) = 0$

$$\therefore \quad \underline{\underline{H}}(\underline{x}_k)(\underline{x}_{k+1} - \underline{x}_k) = -\nabla f(\underline{x}_k)$$

$$\therefore \quad \underline{x}_{k+1} = \underline{x}_k - \underline{\underline{H}}(\underline{x}_k)^{-1}\nabla f(\underline{x}_k)$$

Advantages of Newton's Method

- Rapid convergence if f is well approximated by a quadratic

Disadvantages

- Can oscillate rather than converge if f is not well approximated by a quadratic
- Need to invert Hessian (computationally onerous if there are many control variables)

[30%]

(b) $$\frac{\partial F}{\partial L} = \frac{2}{u}\left[-\frac{2J}{L^3} + \frac{m}{3}\right]$$

$$\frac{\partial^2 F}{\partial L^2} = \frac{12J}{uL^4}$$

$$\frac{\partial F}{\partial u} = -\frac{2}{u^2}\left[\frac{J}{L^2} + \frac{mL}{3} + M_1\right] + M_2$$

$$\frac{\partial^2 F}{\partial u^2} = \frac{4}{u^3}\left[\frac{J}{L^2} + \frac{mL}{3} + M_1\right]$$

$$\frac{\partial^2 F}{\partial L \partial u} = -\frac{2}{u^2}\left[-\frac{2J}{L^3} + \frac{m}{3}\right]$$

3. (b) continued

For the values specified

$$\frac{\partial F}{\partial L} = \frac{2}{u}\left[\frac{-4}{L^3} + 1\right] \qquad \frac{\partial^2 F}{\partial L^2} = \frac{24}{uL^4}$$

$$\frac{\partial F}{\partial u} = -\frac{2}{u^2}\left[\frac{2}{L^2} + L + 8\right] + 5 \qquad \frac{\partial^2 F}{\partial u^2} = \frac{4}{u^3}\left[\frac{2}{L^2} + L + 8\right]$$

$$\frac{\partial^2 F}{\partial L \partial u} = -\frac{2}{u^2}\left[-\frac{4}{L^3} + 1\right]$$

For $L_1 = 1$ m, $u_1 = 1$

$$\frac{\partial F}{\partial L} = -6 \qquad \frac{\partial F}{\partial u} = -17$$

$$\frac{\partial^2 F}{\partial L^2} = 24 \qquad \frac{\partial^2 F}{\partial u^2} = 44 \qquad \frac{\partial^2 F}{\partial L \partial u} = 6$$

$$\therefore \nabla f = \begin{bmatrix} -6 \\ -17 \end{bmatrix} \qquad \underline{\underline{H}} = \begin{bmatrix} 24 & 6 \\ 6 & 44 \end{bmatrix}$$

$$\therefore \underline{\underline{H}}^{-1} = \frac{1}{1020}\begin{bmatrix} 44 & -6 \\ -6 & 24 \end{bmatrix}$$

$$\therefore \begin{bmatrix} L_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{1020}\begin{bmatrix} 44 & -6 \\ -6 & 24 \end{bmatrix}\begin{bmatrix} -6 \\ -17 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 1.159 \\ 1.365 \end{bmatrix}}}$$

[45%]

(c) For unconstrained minimum $\nabla f = 0$ and $\underline{\underline{H}}$ is positive definite

From above $\frac{\partial F}{\partial L} = 0$ when $L^3 = 4$

$$\therefore \underline{\underline{L^* = 1.587 \text{ m}}}$$

and $\frac{\partial F}{\partial u} = 0$ when

$$u^2 = \frac{2}{5}\left[\frac{2}{L^2} + L + 8\right]$$

3. (c) continued

So for $L = 1.587$ $\quad \underline{\underline{u^* = 2.038}}$

For these values $\dfrac{\partial^2 F}{\partial L^2} = 1.857$

$$\frac{\partial^2 F}{\partial u^2} = 4.908$$

$$\frac{\partial^2 F}{\partial L \partial u} = 0$$

$\therefore \ \underline{\underline{H}} = \begin{bmatrix} 1.857 & 0 \\ 0 & 4.908 \end{bmatrix}$ which is by inspection positive definite

$\therefore$ The global minimum is at $L = 1.587$ m
$u = 2.038$

Newton's Method is moving in the right direction but not very fast, implying that F is not well approximated by a quadratic

[25%]

4 (a) Inequalities of the form

$$\underline{a}.\underline{x} \leq b$$

can be made into equality constraints by introducing a new variable x_s such that

$$\underline{a}.\underline{x} + x_s = b$$

Similarly inequalities of the form

$$\underline{a}.\underline{x} \geq b$$

can be made into equality constraints by introducing a new variable x_s such that

$$\underline{a}.\underline{x} - x_s = b$$

[10%]

(b) Profit $= 12x_1 + 10x_2$

hence f (to be minimised) $= -12x_1 - 10x_2$

Constraints:

Availability of C

$$0.4x_1 + 0.5x_2 \leq 100$$

Availability of D

$$0.6x_1 + 0.5x_2 \leq 80$$

Market for A

$$x_1 \leq 70$$

Market for B

$$x_2 \leq 120$$

Hence in standard form

Minimise $f(\underline{x}) = -12x_1 - 10x_2$

Subject to

$$0.4x_1 + 0.5x_2 + x_3 = 100$$
$$0.6x_1 + 0.5x_2 + x_4 = 80$$
$$x_1 + x_5 = 70$$
$$x_2 + x_6 = 120$$

where x_3, x_4, x_5 and x_6 are slack variables

[20%]

4. (c) If $x_1 = x_2 = 0$ then all constraints are active and the slack variables are the initial basic variables. Thus, in canonical form, the initial tableau is

$$\begin{array}{c}
\begin{array}{ccccccc} & & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \end{array}\\
\left[\begin{array}{ccccccc}
0.4 & 0.5 & 1 & 0 & 0 & 0 & 100\\
0.6 & 0.5 & 0 & 1 & 0 & 0 & 80\\
1 & 0 & 0 & 0 & 1 & 0 & 70\\
0 & 1 & 0 & 0 & 0 & 1 & 120\\
-12 & -10 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ (1)\\ (2)\\ (3)\\ (4)\\ (5) \end{array}$$

x_1 is the entering variable (-12 is the most negative reduced cost)

Possible leaving variables

x_3: $100/0.4 = 250$ $\quad$ x_4: $80/0.6 = 133.3$

x_5: $70/1 = 70$

Hence x_5 is the leaving variable

The next tableau in canonical form is

$$\begin{array}{c}
\begin{array}{ccccccc} \Downarrow & & \Downarrow & \Downarrow & & \Downarrow & \end{array}\\
\left[\begin{array}{ccccccc}
0 & 0.5 & 1 & 0 & -0.4 & 0 & 72\\
0 & 0.5 & 0 & 1 & -0.6 & 0 & 38\\
1 & 0 & 0 & 0 & 1 & 0 & 70\\
0 & 1 & 0 & 0 & 0 & 1 & 120\\
0 & -10 & 0 & 0 & 12 & 0 & 840
\end{array}\right]
\end{array}
\begin{array}{l} \\ (6)=(1)-0.4(3)\\ (7)=(2)-0.6(3)\\ (8)=(3)\\ (9)=(4)\\ (10)=(5)+12(3) \end{array}$$

Now x_2 is the entering variable (only negative reduced cost)

Possible leaving variables

x_4: $38/0.5 = 76$ $\quad$ x_3: $72/0.5 = 144$

x_6: $120/1 = 120$

Hence x_4 is the leaving variable

(cont.)

4 (c) continued

The next tableau in canonical form is

$$\begin{array}{c|l}
\begin{bmatrix}
0 & 0 & 1 & -1 & 0.2 & 0 & 34 \\
0 & 1 & 0 & 2 & -1.2 & 0 & 76 \\
1 & 0 & 0 & 0 & 1 & 0 & 70 \\
0 & 0 & 0 & -2 & 1.2 & 1 & 44 \\
0 & 0 & 0 & 20 & 0 & 0 & 1600
\end{bmatrix} &
\begin{array}{l}
(11) = (6) - (7) \\
(12) = 2(7) \\
(13) = (8) \\
(14) = (9) - (12) \\
(15) = (10) + 10(12)
\end{array}
\end{array}$$

As there are no negative reduced costs the minimum has been found.

From (13) x_1 = amount of A = 70 kg

From (12) x_2 = amount of B = 76 kg

From (15) Profit = £1600

[70%]