ENGINEERING TRIPOS PART IIB

---

Tuesday 24 April 2007 9 to 10.30

---

Module 4A8

ENVIRONMENTAL FLUID MECHANICS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachment: 4A8 Data Card (5 pages).*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 A heavy gas is released on a slope of angle $\theta$ to the horizontal and runs down as a thickening layer. You may assume that the flow is homogeneous across the slope and hence the mean flow is essentially two-dimensional. The flow may be considered to be turbulent and the mixing between the gas and the surrounding air is characterised by an entrainment coefficient $\alpha$. At a distance $s$ down the slope the velocity of the flow is uniform across the thickness of the layer and equal to $U$ m s$^{-1}$. The thickness of the layer (normal to the ground) is $h$ m and the mean density of the layer is $\rho$ kg m$^{-3}$. The air outside the layer is still and its density is $\rho_a$ kg m$^{-3}$. The density difference is very small i.e. $(\rho - \rho_a)/\rho_a << 1$ so that $\rho$ may be replaced by $\rho_a$ where appropriate.

(a) Derive the equations governing the behaviour of this flow. [20%]

(b) Using the equations for conservation of volume and mass show that

$$\frac{d((\rho - \rho_a)Uh)}{ds} = 0$$

[20%]

(c) Assuming that the parameters describing the layer have a power-law variation with distance down the slope:

(i) Find the variation of the thickness $h$ with distance down the slope. [20%]

(ii) Find the variation of the density difference $(\rho - \rho_a)$ with distance down the slope. [20%]

(iii) Find the variation of the velocity $U$ with distance down the slope. [20%]

2 A steady free convection flow is set up near the ground on a hot day due to the mean heat flux $Q$ from the ground to the air above. The mean velocity is zero. Measurements show that the length-scale of the largest eddying motions is approximately $L$. The mean temperature of the air is $T$ and it has a heat capacity $c_p$ and a density $\rho$. The acceleration due to gravity is $g$. The enthalpy equation for this particular situation is

$$\lambda \frac{\partial T}{\partial x_3} - \rho c_p \overline{u_3 \theta} = -Q$$

where $\lambda$ is the thermal conductivity of the air, $x_3$ is the vertical direction, $u_3$ is the vertical velocity fluctuation, $\theta$ is the temperature fluctuation and the overbar denotes a time average. The equation for the turbulent kinetic energy is given on the Data Card.

(a) Explain the meaning of the two terms on the left-hand side of this equation and identify the regions of the flow where each one dominates and explain why. [20%]

(b) By considering the dominant terms in the turbulent kinetic energy equation and using the above equation estimate the size of the smallest eddying motion in the turbulent flow. [20%]

(c) This flow is to be computed using Direct Numerical Simulation. Find an expression for the approximate number of grid points required to accurately compute this flow in terms of $Q$, $g$, $L$, $\rho$, $c_p$, $T$ and $\nu$, where $\nu$ is the kinematic viscosity of the air. [20%]

(d) Find an expression for the time-scale of the largest eddying motions in the flow (the large eddy turnover time) in terms of the same variables. [20%]

(e) Estimate the number of time-steps required in the simulation to capture 10 large eddy turnover times as a function of the same variables. [20%]

(TURN OVER

3 (a) What are the types of pollutant emission sources according to geometrical characterisation? Give an example for each type. [5%]

(b) Explain how to include ground effects in the Gaussian Plume Model for pollutant dispersion. [10%]

(c) Consider a line source at a distance $y_o$ metres above the ground level emitting $(Q/L)$ kg of a pollutant per unit time and per unit length. The pollutant spreads along $y$. There is a mean wind at a speed of $U$ m s$^{-1}$ along the $x$-direction. Using the solutions given on the data card,

(i) Show that, when the ground is non-absorbing, the pollutant concentration on the ground varies as

$$\overline{\phi}(x,0) = \frac{A}{\sigma_y} \exp\left(-\frac{y_o^2}{2\sigma_y^2}\right) ,$$

where $A = 2Q/(UL\sqrt{2\pi})$ and $\sigma_y$ is the dispersion coefficient. [20%]

(ii) By taking $\sigma_y = Bx$ where $B$ is a constant, show that the maximum pollutant concentration occurs at $x = y_o/B$ and the maximum concentration is equal to $0.607A/y_o$. [50%]

(d) Calculate the maximum ground concentration if the line source is located at 10 m above the ground level, the pollutant emission rate is 1 kg m$^{-1}$ s$^{-1}$ and the mean wind speed is 10 km hr$^{-1}$ [10%]

(e) What happens to the maximum ground level concentration if the height of the pollutant source is doubled? [5%]

4 (a) Explain the physical significance of the Damköhler number, Da, and its various limits. [20%]

(b) The mean concentration $\overline{\phi}$ of $NO_2$ in the air above a city at 5:00 AM is measured to be $\overline{\phi_o}$ kg m$^{-3}$ and the concentration is considered to be homogeneous in the mean but finite small-scale fluctuations exist. The turbulence in the city atmosphere has a characteristic timescale $T_{turb}$. The weather is clear and still. Thus, one would expect the following photolytic reaction with a rate constant $k$ s$^{-1}$

$$NO_2 \xrightarrow{h\nu} NO + O$$

to occur during the day.

(i) Calculate the time, in terms of $k$, for the mean $NO_2$ concentration to become 5% of its initial value. Take $k$ to be independent of the time of the day. [35%]

(ii) From the modelled variance equation given in the data card, obtain an expression for the variation of $NO_2$ variance, $g$, with time. Sketch the variation of $g/g_o$ with time for $k = 0$, $k >> 1$, and $k < 1$. $g_o$ is the initial variance. [45%]

**END OF PAPER**

# 4A8: Environmental Fluid Mechanics

## DATA CARD
(5 pages)

## Part I: Turbulence and Fluid Mechanics

**Rotating Flows**

Geostrophic Flow

$$-\frac{1}{\rho}\nabla p = 2\underline{\Omega}\times\underline{u}$$

Ekman Layer Flow

$$-2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

OR

$$-2\Omega_z v = \nu\frac{\partial^2 u}{\partial z^2}$$

$$-2\Omega_z v(u_0 - u) = \nu\frac{\partial^2 v}{\partial z^2}$$

*GEOSTROPHIC VELOCITY*

Solution is

$$u = u_0\left[1 - e^{-z/\Delta}\cos\frac{z}{\Delta}\right]$$

$$v = u_0 e^{-z/\Delta}\sin\frac{z}{\Delta}$$

$$\Delta = \left(\frac{\nu}{\Omega_z}\right)^{1/2}$$

**Turbulent Flows – Incompressible**

Continuity Equation

$$\nabla \bullet \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$$

Momentum Equation

$$\rho \frac{D\underline{U}}{Dt} = -\nabla P + \mu \nabla^2 \underline{U} + \underline{F}$$

$$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} + F_i$$

Enthalpy Equation

$$\rho c_p \frac{DT}{Dt} = -k \frac{\partial^2 T}{\partial x_i^2}$$

Reynolds Transformation

$$U_i = \overline{U_i} + u_i \quad \text{etc}$$

Reynolds Stress

$$= -\rho \overline{u_i u_j}$$

Reynolds Heat Flux

$$= -\rho c_p \overline{u_j \theta}$$

Turbulent Kinetic Energy Equation

$$\frac{D}{Dt}\frac{q^2}{2} = -\overline{u_i u_k}\frac{\partial \overline{U_i}}{\partial x_k} - \varepsilon + \frac{\overline{f_i u_i}}{\rho} + \text{ transport terms}$$

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k}\right)} \quad , \quad \frac{q^2}{2} = \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right)$$

In flows with thermally driven motion

$$\frac{f_i u_i}{\rho} = \frac{g}{T} \bullet \overline{\theta u_i}$$

$i$ = Vertical direction

Dissipation of turbulent kinetic energy

$$\varepsilon \approx \frac{u'^3}{\ell}$$

Kolmogorov microscale

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Taylor microscale (λ)

$$\varepsilon = 15\nu \frac{u'^2}{\lambda^2}$$

($\nu$ is the kinematic viscosity)

**Density Influenced Flows**

Atmospheric Boundary Layer

$$\left.\frac{dT}{dz}\right|_{\text{NEUTRAL STABILITY}} = -\frac{g}{C_p} = \left.\frac{dT}{dz}\right|_{\text{DALR}}$$

$$R_i = \frac{g}{T}\frac{\dfrac{dT}{dz} - \left.\dfrac{dT}{dz}\right|_{\text{DALR}}}{\left(\dfrac{dU}{dz}\right)^2} = \text{RICHARDSON NUMBER}$$

Neutral Stability

$$U = \frac{u_*}{\kappa}\ln\frac{z}{z_0}; \qquad \frac{dU}{dz} = \frac{u_*}{\kappa z}$$

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \quad ; \quad \kappa = \text{ von Kármán Constant} = 0.40$$

Non-Neutral Stability

$$\text{L = Monin-Obukhov length} = -\frac{u_*^3}{\kappa\dfrac{g}{T}\dfrac{Q}{\rho c_p}}$$

Q = surface heat flux

$$\frac{dU}{dz} = \frac{u_*}{\kappa z}\left(1 - 15\frac{z}{L}\right)^{-1/4} \qquad \text{Unstable}$$

$$= \frac{u_*}{\kappa z}\left(1 + 4.7\frac{z}{L}\right) \qquad \text{Stable}$$

Buoyant plume for a point source

$$\frac{d}{dz}\left(\pi R^2 w\right) = 2\pi R u_e \qquad \text{(i)}$$

$$\frac{d}{dz}\left(\rho\pi R^2 w\right) = \rho_a 2\pi R u_e \qquad \text{(ii)}$$

$$\frac{d}{dz}\left(\rho\pi R^2 w^2\right) = g\left(\rho_a - \rho\right)\pi R^2 \qquad \text{(iii)}$$

(i) and (iii) give

$$\pi R^2 w\left(\frac{\rho_a - \rho}{\rho_a}\right)g = \text{constant} = F_0 \text{ (buoyancy flux)}$$

$$u_e = \alpha w$$

($\alpha$ = Entrainment coefficient)

Buoyancy Frequency

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} = \frac{g}{T}\frac{dT}{dz}$$

Actually $\frac{g}{T}\left(\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}\right)$

N = Brunt – Väisälä Frequency or Buoyancy Frequency

## Part II: Dispersion of Pollution in the Atmospheric Environment

Transport equation for the mean of the reactive scalar $\phi$:

$$\frac{\partial\overline{\phi}}{\partial t}+\overline{u}_j\frac{\partial\overline{\phi}}{\partial x_j}=\frac{\partial}{\partial x_j}\left(K\frac{\partial\overline{\phi}}{\partial x_j}\right)+\overline{w}$$

Transport equation for the variance of the reactive scalar $\phi$:

$$\frac{\partial g}{\partial t}+\overline{u}_j\frac{\partial g}{\partial x_j}=\frac{\partial}{\partial x_j}\left(K\frac{\partial g}{\partial x_j}\right)+2K\left(\frac{\partial\overline{\phi}}{\partial x_j}\right)^2-\frac{2}{T_{turb}}g+2\overline{\phi'w'}$$

Mean concentration of pollutant after instantaneous release of $Q$ kg at $t$=0:

$$\overline{\phi}(x,y,z,t)=\frac{Q}{8(\pi t)^{3/2}(K_xK_yK_z)^{1/2}}\exp\left[-\frac{1}{4t}\left(\frac{(x-x_0)^2}{K_x}+\frac{(y-y_0)^2}{K_y}+\frac{(z-z_0)^2}{K_z}\right)\right]$$

Gaussian plume spreading in two dimensions from a source at $(0,0,z_0)$ emitting $Q$ kg/s:

$$\overline{\phi}(x,y,z)=\frac{Q}{2\pi}\frac{1}{U\sigma_y\sigma_z}\exp\left[-\left(\frac{y^2}{2\sigma_y^2}+\frac{(z-z_0)^2}{2\sigma_z^2}\right)\right]$$

One-dimensional spreading from line source at $y=0$ emitting $Q/L$ kg/s/m :

$$\overline{\phi}(x,y)=\frac{Q}{UL}\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{y^2}{2\sigma_y^2}\right)$$

Relationship between eddy diffusivity and dispersion coefficient:

$$\sigma^2=2\frac{x}{U}K$$