ENGINEERING TRIPOS PART IIB

Friday 4 May 2007 9.00 to 10.30

Module 4A12

TURBULENCE

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A12 Data Cards

(i) Vortex Dynamics (1 page)

(ii) Turbulence (2 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 (a) In a steady, two-dimensional flow the temperature field, T, is governed by

$$\mathbf{u}\cdot\nabla T = \alpha\nabla^2 T.$$

Show that, when the thermal diffusivity is zero, the temperature field is of the form $T = T(\psi)$, where ψ is the stream function. Hence show that, when α is small but finite, the heat equation takes the approximate form

$$\mathbf{u}\cdot\nabla T = \alpha\nabla\cdot[T'(\psi)\nabla\psi].$$

Now consider the special case where the streamlines are closed. Integrate this equation over a two-dimensional volume bounded by a closed streamline, C, and hence show that

$$\alpha T'(\psi)\oint\nabla\psi\cdot d\mathbf{S} = 0,$$

where $d\mathbf{S}$ is an element of the surface that bounds the two-dimensional volume. [35%]

(b) Deduce that $T'(\psi) = 0$ and give a physical interpretation of this result. [30%]

(c) State the Prandtl-Batchelor theorem and briefly indicate how you might prove it. What is the physical interpretation of this theorem? [35%]

2 A vortex sheet of changing thickness, δ, takes the form

$$\boldsymbol{\omega} = (u_0/\delta)\exp\left[-\frac{x^2}{\delta^2}\right]\hat{e}_z, \qquad \delta = \delta(t),$$

where x, y and z are Cartesian coordinates and u_0 is a measure of the strength of the sheet, which is a constant. The velocity field associated with the vortex sheet is $\mathbf{u} = u_y(x,t)\hat{\mathbf{e}}_y$.

(a) Sketch the velocity and vorticity fields. [10%]

(b) Starting with the vorticity equation, show that $\boldsymbol{\omega}$ is governed by

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \frac{\partial^2 \boldsymbol{\omega}}{\partial x^2}$$

and hence show that $\delta\delta'(t) = 2\nu$. [40%]

(c) Determine $\delta(t)$ and describe physically what is happening to the sheet. Why is u_0 a constant? [20%]

(d) The vortex sheet is now subject to the irrotational straining flow $u_x = -\alpha x$, $u_z = \alpha z$, where α is a positive constant. Describe qualitatively the various processes which now govern the distribution of vorticity. What would you expect to happen to the sheet when $\alpha << \nu/\delta_0^2$ and $\alpha >> \nu/\delta_0^2$, where δ_0 is the initial sheet thickness? [30%]

(TURN OVER

3 (a) Give a practical example and a brief discussion to illustrate the use of the characteristic scales of turbulence that occur in turbulent flows. [30%]

(b) The general equation for turbulent kinetic energy can be written as

$$0 = -\overline{u_i u_k}\frac{\partial \overline{U_i}}{\partial x_k} - \nu\overline{\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k}\right)} + g\beta\overline{\theta\, u_3}$$

when there are no changes with time and the turbulent transport terms are neglected.

On a sunny day the surface of the earth is heated by solar radiation. The heated surface will cause a vertical convective heat flow and turbulent motion that produce a convective boundary layer.

(i) In the convective atmospheric boundary layer with no mean wind the profile of the vertical heat flux can be approximated to be

$$Q = \rho c_p \overline{\theta u_3} = Q_0\left(1 - x_3/h\right)$$

where h is the depth of the boundary layer and x_3 is the vertical co-ordinate. u_3 is the vertical velocity fluctuation and θ is the temperature fluctuation. All other symbols have their usual meaning. Determine how the dissipation rate of turbulent kinetic energy will vary with height. [20%]

(ii) Assuming that the large eddy length scale l is proportional to the depth of the convective boundary layer h, estimate how the turbulent velocity fluctuations will vary with height through the convective boundary layer. [20%]

(iii) If, near the ground, the large eddy length scale is not constant, but increases linearly with height above the surface, estimate how the turbulent velocity fluctuations will vary with height near the ground. Provide a composite sketch of the variation of the turbulent velocity fluctuations through the convective boundary layer. [30%]

4 20 cm^3 of water (density 1000 kgm^{-3}) is rapidly poured from a height of 20 cm into a mug (of diameter 7 cm) of tea. The resulting volume of the tea and added water is 250 cm^3.

(a) Estimate the increase in mechanical energy of the 250 cm^3 volume of the tea. [10%]

(b) What are the initial typical turbulent velocity fluctuations and the initial Reynolds number? (Assume the energy containing eddies have a size comparable to the mug dimension). The kinematic viscosity of the fluid is $1 \times 10^{-6}\ m^2 s^{-1}$. [20%]

(c) Estimate the turbulent dissipation of kinetic energy per unit mass. [10%]

(d) Neglecting the possible effect of boundaries, determine the evolution of the turbulent kinetic energy per unit mass with time. When does the Reynolds number reduce to 100? [30%]

(e) What is the smallest turbulent eddy size, both initially and when the Reynolds number is 100? [20%]

(f) What is the increase in temperature due to the dissipation of the turbulent kinetic energy? The specific heat capacity of the fluid is 4200 $J\ kg^{-1}\ K^{-1}$. [10%]

END OF PAPER

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot dS = \oint A \cdot dl$

Vector Identities

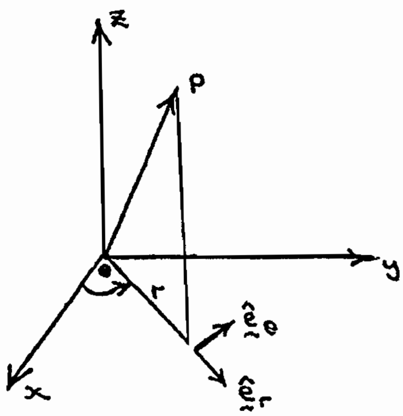
$\nabla (A \cdot B) = (A \cdot \nabla) B + (B . \nabla) A + A \times (\nabla \times B) + B \times (\nabla \times A)$
$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$
$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla) A - (A \cdot \nabla) B$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times A) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & \quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'} \end{aligned}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi' \phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - \overline{2\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= (\nu/\varepsilon)^{1/2} \\ v_K &= (\nu\varepsilon)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$