ENGINEERING TRIPOS PART IIB

Friday 27 April 2007 2.30 to 4

Module 4C8

APPLICATIONS OF DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4C8 datasheet (3 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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1 It has been suggested that the performance of a car can be improved if the suspension geometry of the rear wheels causes them to adopt a steering angle ε proportional to the lateral force Y_r on each tyre. A 'bicycle' model of the car, as shown in Fig. 1 is used to investigate the effects of this passive rear wheel steering. In the model, which has the usual notation, the front wheels are steered at angle δ and the rear wheels adopt angle ε, where $\varepsilon = \gamma Y_r$ and γ is a constant. All angles should be considered to be small.

(a) Assuming a lateral creep coefficient C for the combined responses of the tyres on each axle, show that the lateral forces Y_f and Y_r are given by

$$Y_f = C\left(\frac{v + a\Omega}{u} - \delta\right) \quad \text{and} \quad Y_r = \frac{C}{u}\left(\frac{v - b\Omega}{1 + \gamma C}\right)$$ [30%]

(b) Derive equations of motion for the response of the vehicle to side force Y applied to the centre of mass, and steer angle δ. Put these equations into matrix form

$$[M]\dot{\mathbf{y}} + [K]\mathbf{y} = \mathbf{F}, \quad \text{where} \quad \mathbf{y} = [v \quad \Omega]^{\mathrm{T}}$$

Define $[M]$, $[K]$ and $\mathbf{F}$. [30%]

(c) Without solving the equations of motion, explain how you would determine the following vehicle responses:

(i) the conditions for stable motion in a straight line;

(ii) the steady-state yaw rate of the vehicle when it is subjected to a lateral force Y at the centre of mass;

(iii) the steady-state cornering behaviour, when a constant steer angle δ is applied and held;

(iv) the transfer function relating the sideslip response $\beta = v/u$ to a sinusoidal steer angle $\delta = \Delta \sin \omega t$.

[40%]

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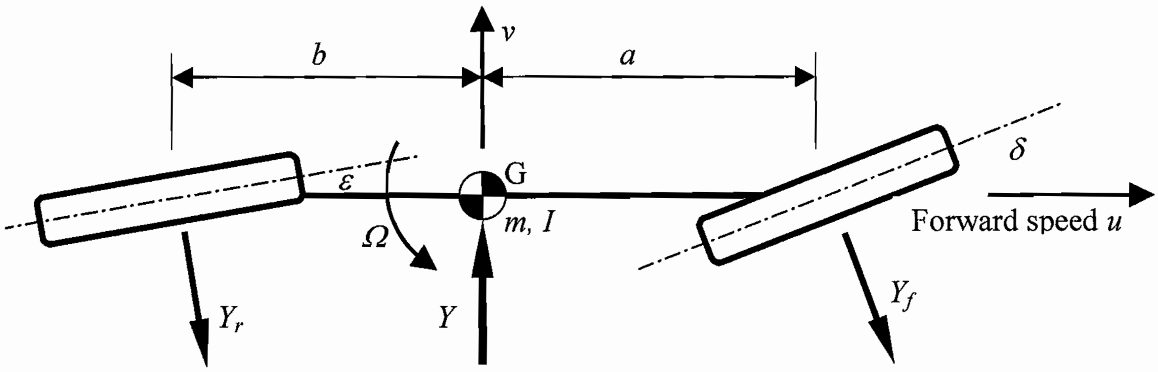

Fig. 1

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2 Figure 2 shows a special railway vehicle that is intended for measuring the vertical roughness of the track. The centre wheelset is located midway between the front and rear wheelsets of the vehicle, which are distance $2L$ apart. It is attached to the vehicle body through a suspension system which has lateral stiffness k_1 (restoring force per unit displacement) and yaw stiffness k_2 (restoring torque per unit of yaw angle). A transducer monitors the vertical displacement of the centre wheelset relative to the vehicle body. It is desired to design the suspension system of the centre wheelset to minimise its lateral tracking error in curves.

(a) Show that in a turn of constant radius R at steady speed u the creep forces on the centre wheelset result in a total lateral force Y of

$$Y = 2C\left(\frac{\dot{y}}{u} + \theta\right)$$

and a total yawing moment N of

$$N = 2dC\left(\frac{\varepsilon y}{r} - \frac{d\dot{\theta}}{u} - \frac{d}{R}\right)$$

where y is the lateral tracking error of the wheelset, θ is the yaw angle of the wheelset, $2d$ is the track gauge, r is the rolling radius of each wheel when $y = \theta = 0$, ε is the effective conicity of the wheelset and C is the creep coefficient relating the creep velocities to the creep forces. State your assumptions.

[50%]

(b) The vehicle is moving around a curve of radius R in steady motion ($\dot{y} = \dot{\theta} = 0$). Assuming that the front and rear wheelsets track with zero lateral tracking error, show that with a suitable choice of the relationship between the stiffnesses k_1 and k_2 the lateral tracking error of the centre wheelset y can be eliminated entirely. What is then the angle θ?

[50%]

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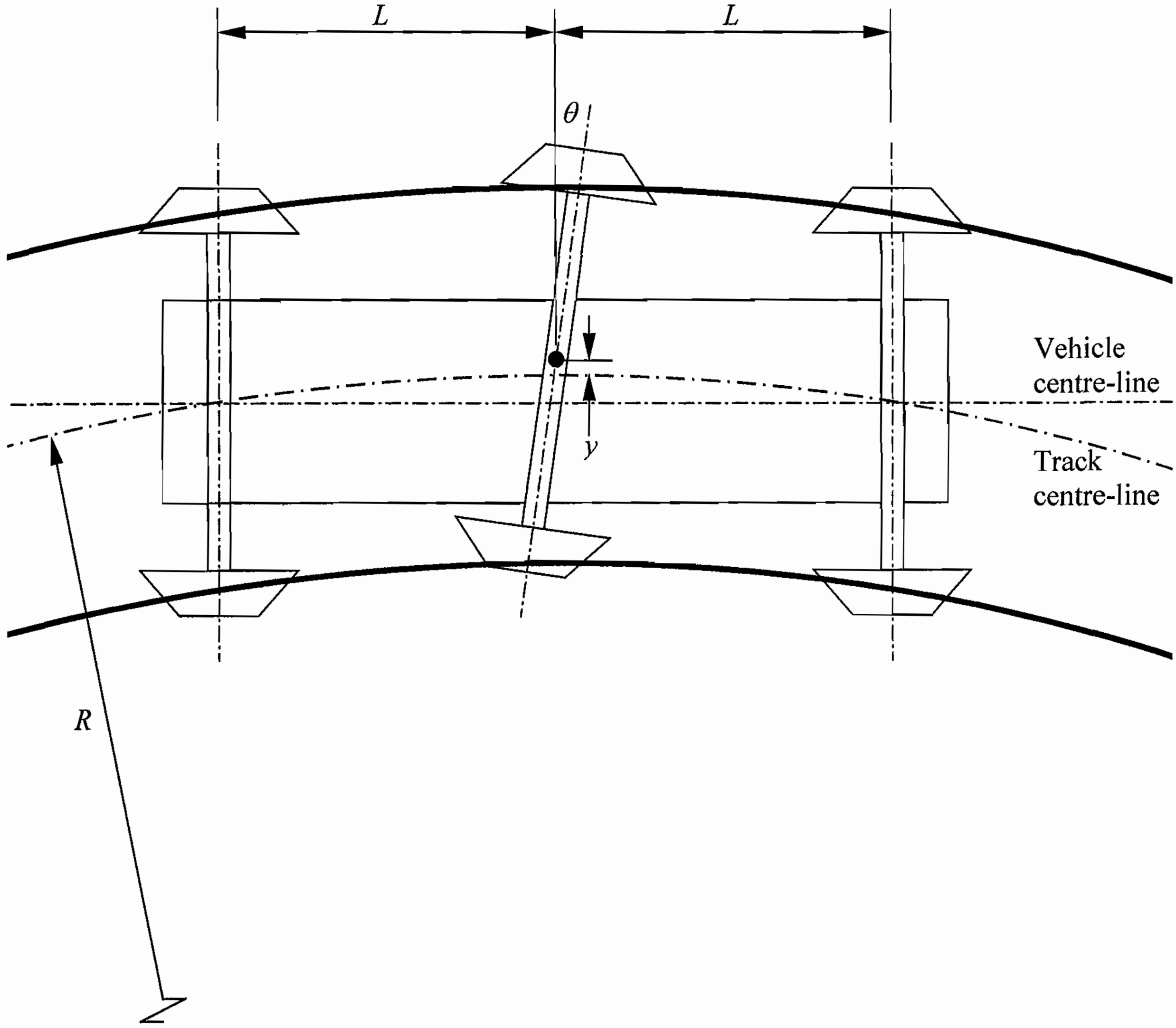

Fig. 2

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3 (a) The formula for the gravitational potential at a point $\mathbf{r}$ caused by a point mass M at the origin is given by

$$U(\mathbf{r}) = \frac{GM}{|\mathbf{r}|}$$

Show that this formula is consistent with Newton's formula for the gravitational force between two bodies, and use your result to prove that a body with distributed mass having density $\rho(\mathbf{r})$ will satisfy Poisson's equation, namely

$$\nabla^2 U = -4\pi G\rho$$

[30%]

(b) Laplace's equation in spherical co-ordinates can be written as

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial U}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0$$

By considering a solution in the form $U = R(r)T(\theta)$, show that solutions to Laplace's equation can be found by solving

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \quad \text{and} \quad \frac{d^2T}{d\theta^2} + \cot\theta\frac{dT}{d\theta} + n(n+1)T = 0$$

where n is a constant. What are the solutions for r, for integer values of n? [30%]

(c) Without going into further mathematical detail, explain how these equations lead to the approximate expression

$$U(r,\theta) = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right\}$$

for the external potential of the earth. Give brief explanations for:

(i) the lack of a term involving $n = 1$ in the expression above;

(ii) the feature of the earth which is modelled by the $n = 2$ term;

(iii) the validity of assuming that U is not a function of ϕ. [20%]

(d) GPS satellites orbit the earth in near-circular orbits with an inclination of 55°. Describe two ways in which the $n = 2$ and higher terms in the expression above cause the orbital parameters of GPS satellites to vary over time, stating which parameters are altered by each effect. [20%]

4 (a) Show that Newton's equations of orbital motion, namely

$$r^2\dot{\theta} = h \quad \text{and} \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

where h and μ are positive constants, can be reduced to

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

by making the substitution $u = 1/r$. What is the physical significance of the constants h and μ? [40%]

(b) Write down a general solution for the differential equation for u, and show how this can be reduced to the standard equation for a Keplerian orbit, namely

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

where a and e are constants whose meaning should be explained. [30%]

(c) Use your results to prove Kepler's third law, namely that the period of a Keplerian orbit is independent of h and e, but proportional to $a^{3/2}$.

[30%]

END OF PAPER

Engineering Tripos Part IIB

Data sheet for Module 4C8: Applications of Dynamics

DATA ON VEHICLE DYNAMICS

1. Creep Forces In Rolling Contact

1.1 Surface tractors

Longitudinal force $X = \iint_A \sigma_x \, dA$

Lateral force $Y = \iint_A \sigma_y \, dA$

Realigning Moment $N = \iint_A \left(x\,\sigma_y - y\,\sigma_x \right) dA$

where

σ_x, σ_y = longitudinal, lateral surface tractions

x, y = coordinates along, across contact patch

A = area of contact patch

1.2 Brush model

$\sigma_x = K_x q_x, \ \sigma_y = K_y q_y$ for $\sqrt{\sigma_x^2 + \sigma_y^2} \le \mu p$

where

q_x, q_y = longitudinal, lateral displacements of 'bristles' relative to wheel rim

K_x, K_y = longitudinal, lateral stiffness per unit area

μ = coefficient of friction

p = local contact pressure

1.3 Linear creep equations

$X = -C_{11}\xi$

$Y = -C_{22}\alpha - C_{23}\psi$

$N = C_{32}\alpha - C_{33}\psi$

where X, Y, N, are defined as in 1.1 above.

C_{ij} = coefficients of linear creep

ξ = longitudinal creep ratio = longitudinal creep speed/forward speed

α = lateral creep ratio = (lateral speed /forward speed) - steer angle

ψ = spin creep ratio = spin angular velocity/forward speed

2. Plane Motion in a Moving Coordinate Frame

$$\ddot{\mathbf{R}}_{O_1} = (\dot{u} - v\Omega)\mathbf{i} + (\dot{v} + u\Omega)\mathbf{j}$$

$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ axis system fixed to body at point O_1

where

u = speed of point O_1 in $\mathbf{i}$ direction

v = speed of point O_1 in $\mathbf{j}$ direction

$\Omega\mathbf{k}$ = absolute angular velocity of body

3. Routh-Hurwitz stability criteria

$$\left(a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if all $a_i > 0$

$$\left(a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 > a_0 a_3$

$$\left(a_4\frac{d^4}{dt^4} + a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

DATA ON POTENTIAL THEORY AND ORBITS

1 For a distribution of mass with density $\rho(\mathbf{r})$ the gravitational potential U satisfies Poisson's equation

$$\nabla^2 U = -4\pi G\rho$$

where G is the gravitational constant (= 6.67×10^{-11} $\mathrm{Nm^2kg^{-2}}$). The gravitational force $\mathbf{F}$ experienced by a unit mass is given by

$$\mathbf{F} = \nabla U .$$

2 *In vacuo* $\rho = 0$, so that U satisfies Laplace's equation

$$\nabla^2 U = 0 .$$

3 For a point mass M at the origin

$$U(\mathbf{r}) = GM/|\mathbf{r}| .$$

For a general distribution of matter

$$U(\mathbf{r}) = G\iiint \frac{\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}}{|\mathbf{r}-\mathbf{x}|} .$$

For a thin spherical shell of radius a and mass dM

$$U(r) = \begin{cases} GdM/|r|, & r > a \\ GdM/a, & r < a \end{cases}$$

4 Equations of motion for a particle in a plane orbit, in plane polar coordinates (r,θ):

$$\ddot{r} - r\dot{\theta}^2 = f_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = f_\theta$$

where f_r, f_θ are the radial and transverse force components, per unit mass.

If $f_\theta = 0$ (i.e. for a central force) the second equation leads to conservation of angular momentum:

$$r^2\dot{\theta} = h = constant .$$

5 For a central force, the substitution $u = 1/r$ leads to an equation for the *shape* of the orbit, expressed as $u = u(\theta)$. The central force (assumed attractive) is described by a function $f(u)$ per unit mass, and for a given angular momentum per unit mass h the orbit satisfies

$$\frac{\partial^2 u}{\partial\theta^2} + u = \frac{f(u)}{h^2u^2} .$$

6 The equation of an ellipse in polar coordinates (r,θ) relative to a focus is

$$r = \frac{L}{(1+e\cos\theta)}$$

where e is the eccentricity.

The semi-major axis is $a = L/(1-e^2)$, the semi-minor axis is $b = L/\sqrt{1-e^2}$.

7 Spherical polar coordinates. Define (r, θ, ϕ) so that r is radial distance, θ is angle from the polar axis (co-latitude) and ϕ is the angle of longitude. Then:

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial U}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

8 Axisymmetric solutions to Laplace's equation arising from separation of variables in spherical polar coordinates are

$$U(r,\theta) = \begin{cases} r^n P_n(\cos\theta) \\ r^{-n-1} P_n(\cos\theta) \end{cases}$$

where P_n is the Legendre polynomial of order n, describing the nth zonal harmonic. The first few Legendre polynomials are as follows:

$$P_0(\xi) = 1 \qquad P_1(\xi) = \xi \qquad P_2(\xi) = (3\xi^2 - 1)/2$$

$$P_3(\xi) = (5\xi^3 - 3\xi)/2 \qquad P_4(\xi) = (35\xi^4 - 30\xi^2 + 3)/8 \,.$$

9 The external potential of the Earth can be expressed as a sum of spherical-harmonic contributions. We consider in detail only the effect of the zonal harmonics, whose contribution can be written in standard form

$$U(r,\theta) = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}(R/r)^n J_n P_n(\cos\theta)\right].$$

For the Earth, $\mu = G \times M_{Earth} = 398603$ km^{3}s^{-2}, mean radius $R = 6378$ km,

$$J_2 = 1082\times10^{-6}, \quad J_3 = -2.55\times10^{-6}, \quad J_4 = -1.65\times10^{-6}$$

Gravitational mass of the sun = 332946μ, gravitational mass of the moon = μ/81.3

Mean radius of Earth's orbit = 1.496×10^8 km, that of moon's orbit = 384400 km.