ENGINEERING TRIPOS PART IIB

Wednesday 25 April 2007 2.30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than **two** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

Attachments:
Special datasheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) If $a_i = e_{ijk} b_j c_k$ and $b_i = e_{ijk} g_j h_k$ show that

$$a_i = g_k c_k h_i - h_k c_k g_i$$

and write this expression in its corresponding vector formulation. [50%]

(b) For an elastic-perfectly plastic solid satisfying Drucker's postulates:

(i) define the limit stress state; [15%]

(ii) show that $\dot{\sigma}_{ij} = 0$ at the limit stress state. [35%]

2 A material has a uniaxial stress-strain curve described by the relation $\sigma = A\varepsilon^n$ where A and n are material constants. It can be assumed that the material does not deform elastically, is incompressible and yields according to the von Mises criterion.

(a) Show that plastic deformation of the material follows the relation

$$\dot{\varepsilon}_{ij}^p = \frac{3}{2}\frac{\dot{\varepsilon}_e^p}{\sigma_e} s_{ij}$$

where s_{ij} is the deviatoric stress tensor. Quantities $\varepsilon_e^p = \sqrt{(2/3)\varepsilon_{ij}^p \varepsilon_{ij}^p}$ and $\sigma_e = \sqrt{(3/2) s_{ij} s_{ij}}$ are respectively the effective plastic strain and the effective stress. [30%]

(b) A thin-walled cylinder with closed ends is constructed of this material. The cylinder has initial radius r_0 and wall thickness t_0 and is subject to an internal pressure p. Show that the radial strain ε_r of the cylinder is related to ε_e^p by the relation

$$\varepsilon_r = -\frac{\sqrt{3}}{2}\varepsilon_e^p$$ [30%]

(c) Show that the wall thickness of the cylinder changes with loading according to the relation

$$t = t_0 \exp\left(-\frac{\sqrt{3}}{2}\varepsilon_e^p\right)$$ [15%]

(d) For $A = 800$ MPa, $n = 0.25$, $r_0 = 200$ mm and $t_0 = 4$ mm find the decrease in wall thickness when the internal pressure p is equal to 10 MPa. [25%]

3 A circular disc of linear elastic material with radius a and height h is compressed between two flat, parallel platens. The surfaces are such that there is no slip between the platens and the disc as it deforms.

(a) If the deformation field within the disc can be assumed to be axisymmetric and of the form

$$u_r = \frac{3Bk}{h^2}(h-z)rz \quad \text{and} \quad u_z = -Bk\left(\frac{3z^2}{h} - \frac{2z^3}{h^2}\right) - Bz$$

in which B and k are arbitrary constants, derive expressions for the strain field, i.e. for the quantities $\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ and γ_{rz}. If $B > 0$ what can you deduce from the dilation e, defined as ε_{ii}, about the numerical value of the Poisson's ratio ν? [30%]

(b) When the thickness of the disc has fallen from h to $h-\delta$ what is the relation between δ, h, B and k? At this instant the axial load applied to the disc is of magnitude P. Explain why the work done V in compressing the disc is of magnitude $P\delta/2$. If the effective modulus of the material of the disc, defined as axial stress ÷ axial strain, is E_a then find an expression for V in terms of E_a, a, δ and h. [30%]

(c) The local strain energy density at any point in the material can be written as

$$\frac{1}{2}\left[\lambda e^2 + 2\mu\left(\varepsilon_r^{\,2} + \varepsilon_\theta^{\,2} + \varepsilon_z^{\,2}\right) + \mu\gamma_{rz}^{\,2}\right]$$

where λ and μ are the Lamé elastic constants. By a suitable scheme this can be integrated over the volume of the block to give a second estimate of V, specifically

$$V = 2\mu\pi a^2 hB^2\left[\frac{q+1}{2} + Zk^2 + k\right] \quad \text{where} \quad q = \frac{\nu}{1-2\nu} \quad \text{and} \quad Z = \frac{9}{10} + \frac{3}{8}(a/h)^2$$

The two expressions for V can be equated to provide an estimate for the ratio E_a/E in terms of the variable k. If k is chosen to minimise the ratio E_a/E will this provide an upper or a lower bound on E_a? [25%]

(d) The resulting expression E_a/E for relatively thin discs can be written as

$$\frac{E_a}{E} \simeq \frac{1}{1+\nu}\left[\frac{2Zq}{2Z+q}\right]$$

When $\nu \to 0.5$ so that $q \gg Z$ find an *approximate* relation between E_a/E and the aspect ratio a/h. [15%]

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$	$a_i e_i$
$\underset{\sim}{a} \bullet \underset{\sim}{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b}$	$c_i = e_{ijk} a_j b_k$
$\underset{\sim}{d} = \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	$e_{ijk} = 1$ when indices cyclic; $= -1$ when indices anticyclic and $= 0$ when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$
trace$a$	tr$a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
gradϕ $= \nabla \phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underset{\sim}{V}$	$V_{i,i}$
curl$\underset{\sim}{V} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{V}$	$e_{ijk} V_{k,j}$

Rotation of Orthogonal Axes

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$$

$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

$$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$$

$s^3 - I_1' s^2 + I_2' s - I_3' = 0$

$$I_1' = s_{ii} = \mathrm{tr}s \; ; \; I_2' = \frac{1}{2}s_{ij}s_{ij} \; ; \; I_3' = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

equilibrium $\quad \sigma_{ij,i} + b_j = 0$

small strains $\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

compatibility $\quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$

equivalent to $\quad e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\dfrac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $\quad E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\quad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ii} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ii} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi_{,2} = \frac{\partial\phi}{\partial x_2} \; , \; \sigma_{32} = -\phi_{,1} = -\frac{\partial\phi}{\partial x_1} \; \text{ and } \; T = 2\iint_A \phi(x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\frac{E}{2(1+\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\frac{E-2G}{2G}$	–	$\frac{(2G-E)G}{E-3G}$
E, λ	–	$\frac{E-\lambda+R}{4\lambda}$	$\frac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\frac{2G\nu}{1-2\nu}$
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\frac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$		–

$R=\sqrt{E^2+2E\lambda+9\lambda^2}$

JAW