ENGINEERING TRIPOS PART IIB

Wednesday 25 April 2007 2.30 to 4

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Formulae sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed
Supplementary pages: One extra copy of Fig. 1 (Question 1) and two extra copies of Fig. 3 (Question 3).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) $F(s)$ is a conformal mapping at a given point if it is analytic and has nonzero derivative at the point. Conformal mappings preserve angles and their sense.

Applications of conformal mappings: (1) an understanding of the Nyquist stability criterion in terms of anxious crossings, (2) calculation of breakaway points in root-locus diagrams, (3) calculation of the sense of rotation of mappings of small semicircular arcs in Nyquist diagrams, (4) calculation of approximate locations of closed-loop poles near to the imaginary axis in Nyquist diagrams.

(b) The actual plant used for this example was $(s+1)^3/((0.4s+1)s^3)$.

(i) The Nyquist diagram (not to scale) for $G(s)$ can be found on the right of Figure 1. On the left is the Nyquist D-contour.

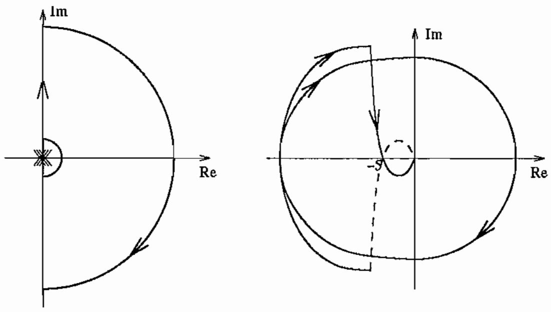

Fig. 1

Note that the small semi-circle gets mapped into an arc passing through 3×180^o.

If $-\infty < -1/k < -5$ (or $0 < k < 0.2$) then there are 2 clockwise encirclements which means the feedback system will have 2 RHP poles.

If $-5 < -1/k < 0$ (or $0.2 < k < \infty$) then there are 0 encirclements which means the feedback system is stable.

If $0 < -1/k < \infty$ (or $k < 0$) then there is 1 clockwise encirclement which means the feedback system will have 1 RHP pole. [25%]

(ii) The actual closed-loop dominant poles are: $0.04 \pm j0.62$ ($k = 1/7$), $0.00 \pm j0.71$ ($k = 1/5$), $0.10 \pm j0.85$ ($k = 1/3$). The estimated location of the closed-loop poles near the imaginary axis can be found attached. [25%]

(iii) A circle centred at -1 and of radius 1 does not intersect, but just touches, $G(j\omega)$. Hence the minimum value of $|1 + G(j\omega)|$ over all frequencies equals 1. Hence the maximum value of $|S(j\omega)|$ over all frequencies equals 1. [20%]

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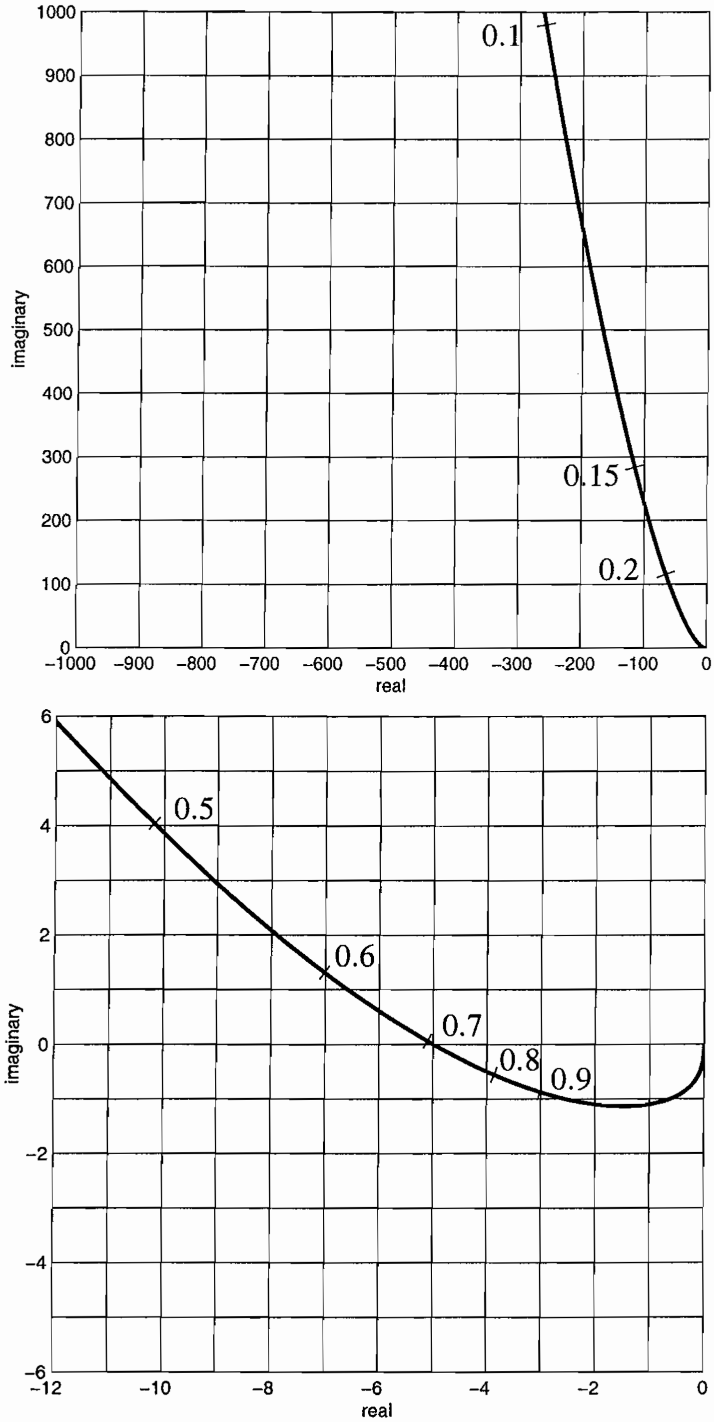

Fig. 1

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2 (a) Briefly describe reasons for the use of feedback control in engineering systems. What are the disadvantages of feedback? [25%]

(b) Suppose $F(s)$ is a rational transfer-function which is stable and fixed. Let $f(\omega)$ be a positive, continuous function. The feedback configuration of Fig. 2 is said to be robustly stable if it is stable for all stable $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq f(\omega)$ (for all ω). The Small Gain Theorem states that Fig. 2 is robustly stable if and only if:

$$|F(j\omega)| < \frac{1}{f(\omega)} \text{ for all } \omega. \tag{1}$$

(i) Show that (1) is a sufficient condition for robust stability. [15%]

(ii) Let $F(s) = 1/(2s+1)$ and $f(\omega) = \sqrt{\omega^2+1}$. Find a $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq f(\omega)$ (for all ω) for which the feedback configuration of Fig. 2 is unstable. [20%]

(c) Let $G_1(s) = G(s) + \Delta(s)$ where $G(s)$ is fixed and known and $\Delta(s)$ is known only to be stable and satisfy $|\Delta(j\omega)| \leq h(\omega)$ (for all ω) for some positive continuous function $h(\omega)$.

(i) Find a necessary and sufficient condition for a controller $K(s)$ to stabilise all such $G_1(s)$. [20%]

(ii) Let

$$G_1(s) = \frac{1}{s} + \frac{e^{-sT}}{s+a}$$

where it is known only that $T > 0$ and $a > 0$. Suppose the controller $K(s) = 1$ is chosen in the standard unity negative feedback arrangement. For what values of T and a can closed loop stability be guaranteed? [20%]

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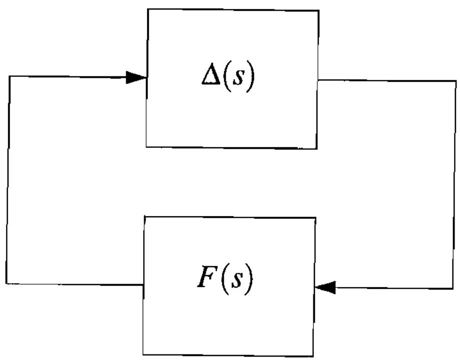

Fig. 2

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3 Fig. 3 is the Bode diagram of a system $G(s)$ for which a feedback compensator $K(s)$ is to be designed. It may be assumed that $G(s)$ is a real-rational transfer function, and that all poles and zeros have moduli which lie within the range of frequencies shown on the diagram.

(a) (i) Sketch on a copy of Fig. 3 the expected phase of $G(j\omega)$ if $G(s)$ were stable and minimum phase. [10%]

(ii) Determine whether $G(s)$ has any right half plane poles or zeros (it doesn't have both), and estimate their location (if there are any). [10%]

(iii) Comment on any limitations that this might impose on the achievable crossover frequency. [10%]

(b) A feedback compensator $K(s)$ is to be designed to simultaneously satisfy the following specifications:

A: internal stability of the closed-loop,

B: a steady-state error of 0.1 for a unit ramp reference input,

C: $|G(j\omega)K(j\omega)| = 1$ at $\omega = 4$ rad/sec,

D: a phase margin of at least 40°.

(i) Explain why these specifications cannot be met if $K(s)$ has just one pole and one zero. [Hint: you may find it helpful to break the argument down into two cases where $K(s)$ is a lag compensator or a lead compensator, respectively.] [35%]

(ii) Find a controller consisting of a lead compensator and a lag compensator together to achieve the desired specifications. [Hint: you may find it helpful to design the lead compensator first to slightly exceed the required phase margin.] Show on another copy of Fig. 3 the effect of this compensator on the return-ratio transfer function. [35%]

Two copies of Fig. 3 are provided on separate sheets. These should be handed in with your answers.

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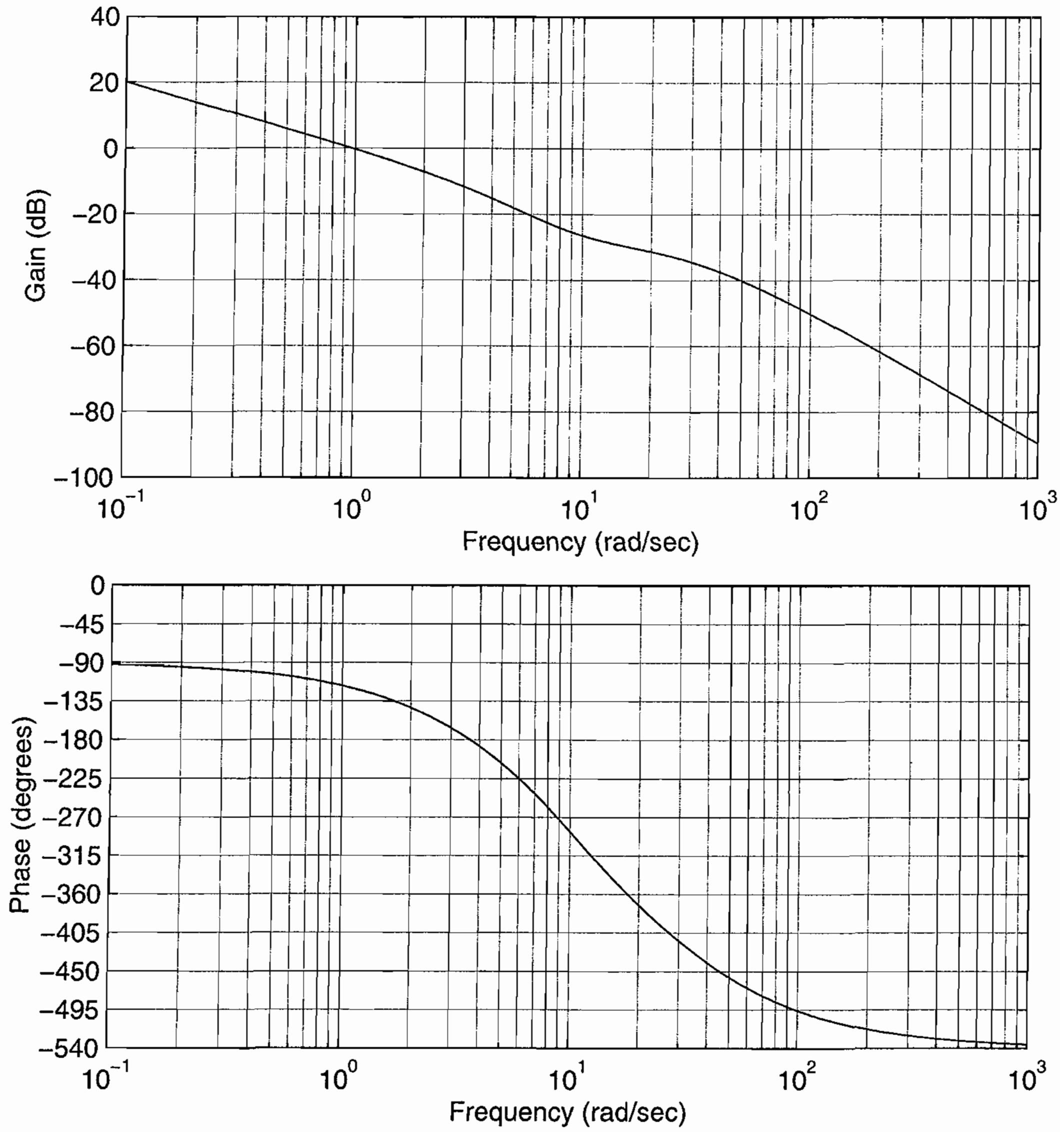

Fig. 3

END OF PAPER

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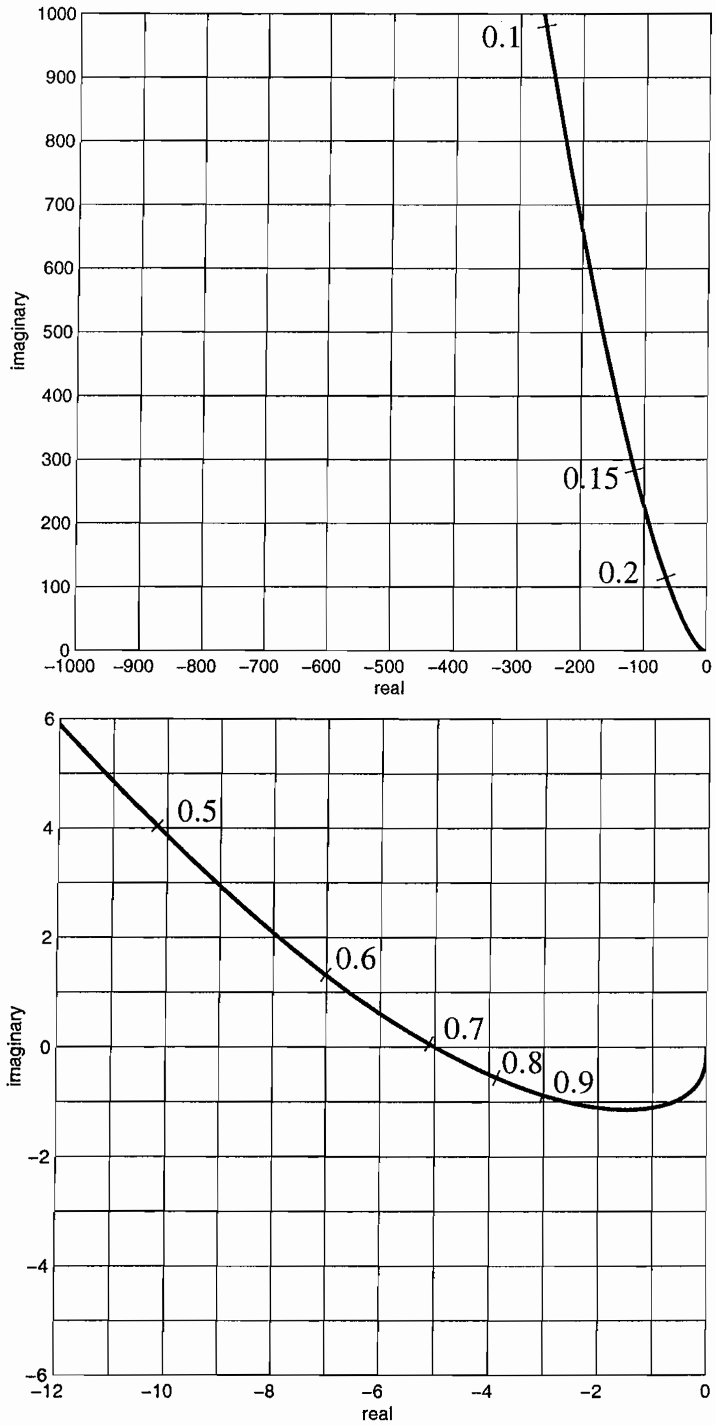
Wednesday 25 April 2007, Module 4F1, Question 1.



Extra copy of Fig. 1: Frequency response of $G(s)$ for Question 1.

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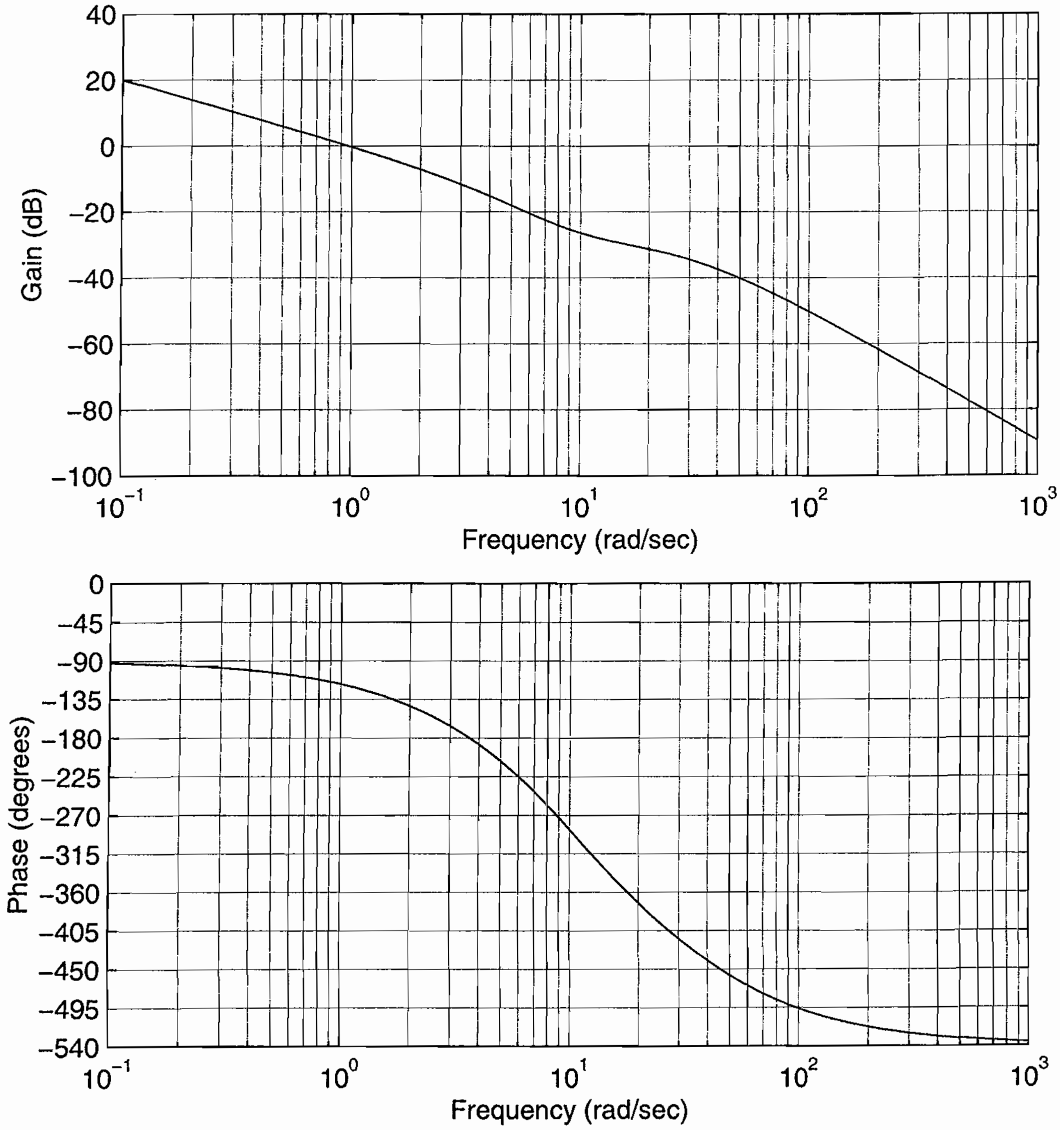
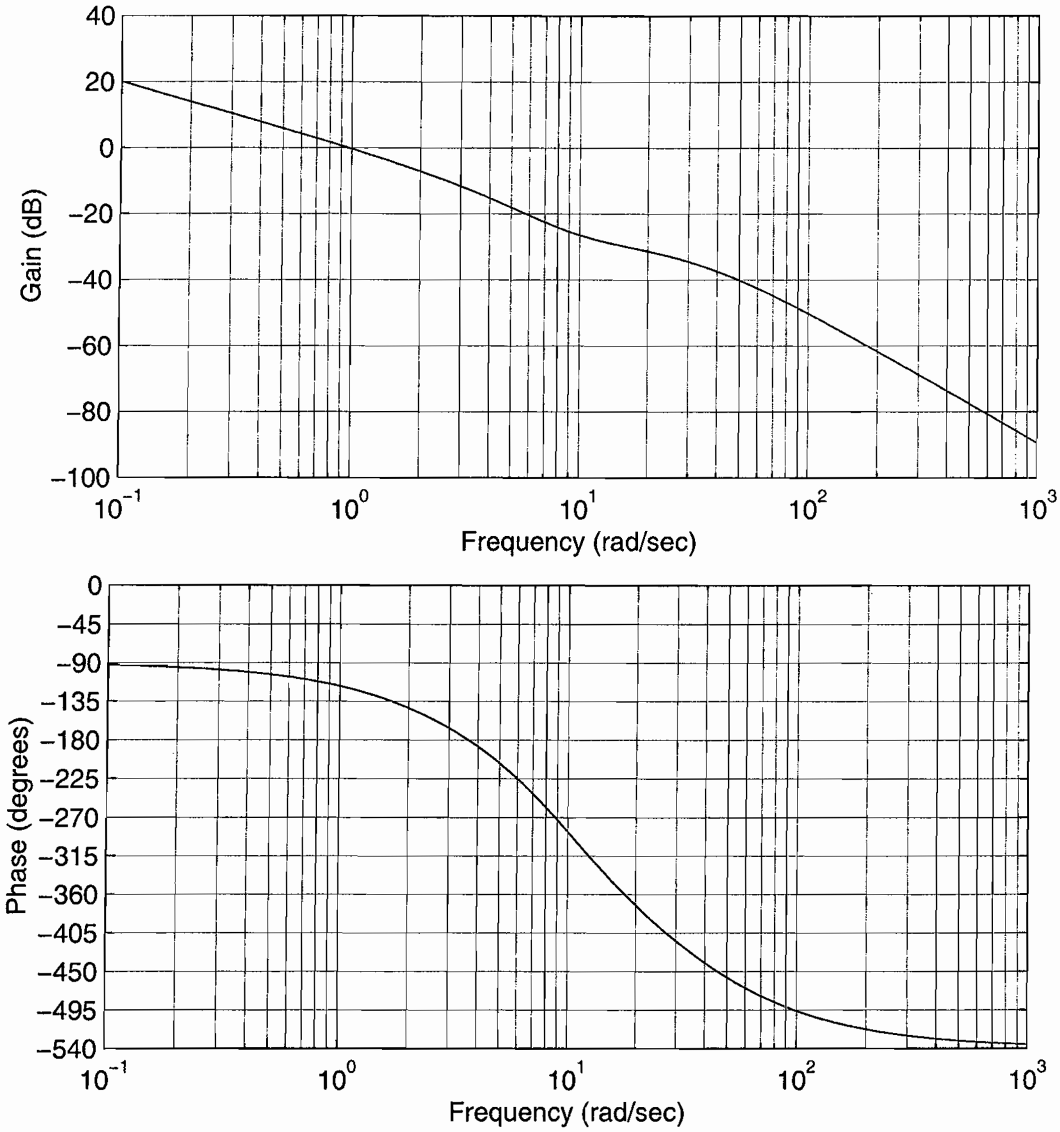
Wednesday 25 April 2007, Module 4F1, Question 3.



Extra copy of Fig. 3: Bode diagram of $G(s)$ for Question 3.

ENGINEERING TRIPOS PART IIB

Wednesday 25 April 2007, Module 4F1, Question 3.



Extra copy of Fig. 3: Bode diagram of $G(s)$ for Question 3.

UNIVERSITY OF CAMBRIDGE

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

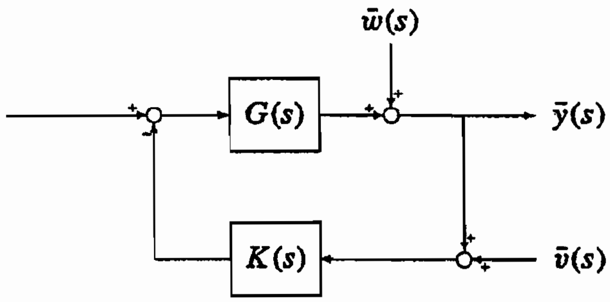
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2\arctan\alpha - 90^\circ.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^v\big)\right| dv}$$

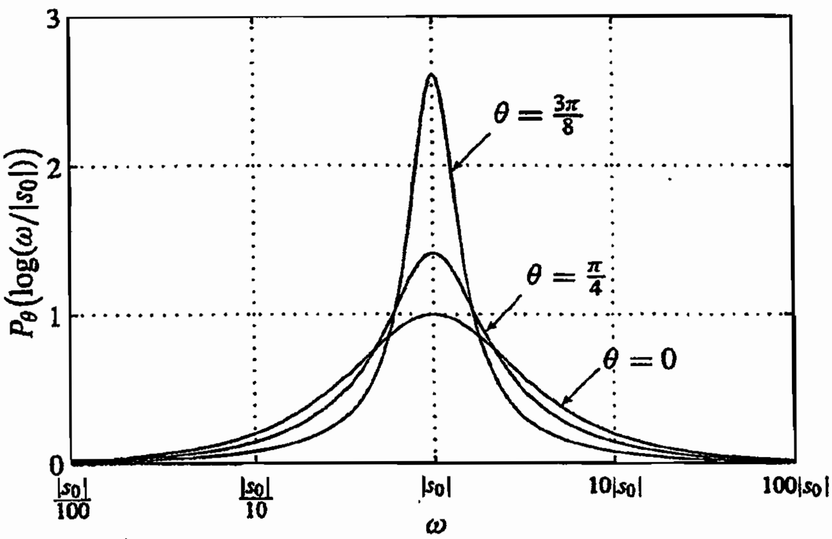
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith
November 2002