ENGINEERING TRIPOS PART IIB

Wednesday 2 May 2007 2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the closed-loop system of Fig. 1.

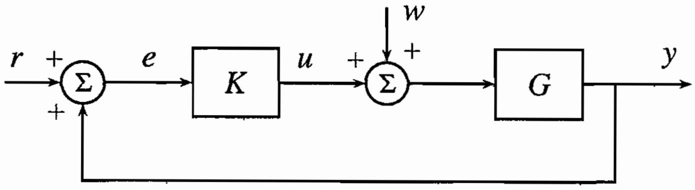

Fig. 1

(a) Show that the closed loop transfer function from $\begin{bmatrix} w \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is given by

$$\begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} G & I \end{bmatrix}$$

[10%]

(b) Let

$$G(s) = \frac{1}{s-2}$$

and $K(s) = k$ in the arrangement of Fig. 1. Recall that the stability margin $b(G,k)$ is defined as the inverse of the $\mathcal{H}_\infty$ norm of the transfer function found in part (a). Calculate $b(G,k)$ as a function of k, for $k < -2$. [30%]

Hint: you may find the following identities useful: $\sigma_i^2(XYZ) = \lambda_i(XYZZ^*Y^*X^*) = \lambda_i(YZZ^*Y^*X^*X)$.

(c) Find k_{opt}, the value of k that maximises the stability margin in part (b), and compute $b(G, k_{\text{opt}})$. Hence, find a positive lower bound on *emax*, the maximum stability margin. [20%]

(d) Explain briefly what is meant by a *normalized coprime factorization*. Find a normalised coprime factorisation of $G(s)$ above in the form
$N(s) = 1/(s+\lambda), \quad M(s) = (s-2)/(s+\lambda)$. [20%]

(e) Let

$$G_1(s) = \frac{1}{s - 2 + 0.1(s + \sqrt{5})\exp(-s)}$$

Find an upper bound on the gap between G and G_1. Is it possible to conclude from this, and your answer to part (c), whether or not G_1 is stabilised by $K(s) = k_{\text{opt}}$? [20%]

2 (a) Consider an optimal control problem for the plant

$$\dot{x} = Ax + Bu$$

where the objective is to minimize

$$J = \int_0^T u^T u\, dt + \lambda^2 x(T)^T x(T)$$

Write down the Hamilton-Jacobi-Bellman equation for this problem and show that the solution is of the form $V(x,t) = x^T Y^{-1} x$ where Y solves the Riccati equation

$$\dot{Y} = AY + YA^T - BB^T, \quad Y(T) = \frac{1}{\lambda^2} I$$

Hint: as part of your derivation you will need to show that $\frac{d}{dt}Y^{-1} = -Y^{-1}\frac{dY}{dt}Y^{-1}$. [30%]

(b) A system satisfies the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

with initial conditions $x_1(0) = L$, $x_2(0) = 0$. Consider the problem of bringing the system to rest at a fixed time T whilst minimizing the cost

$$J = \int_0^T \frac{1}{2}(u^2)\, dt$$

Use your results to part (a) to show that the optimal cost has the value $12\frac{L^2}{T^3}$. [30%]

(c) The same system as in part (b) is again to be brought to rest at the origin whilst minimizing the cost

$$J = \int_0^T \frac{1}{2}(k^2 + u^2)\, dt$$

where the time T is now unconstrained. Use your results to part (c) to find, in terms of k, the time T at which the system reaches the origin under optimal control. [30%]

(d) Give a physical interpretation of the problem in part (c), and comment on the solution. [10%]

(TURN OVER

3 Consider a generalized plant $P(s)$ with state-space realization

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} x + u$$

$$y = x$$

where u is the control input and w is a disturbance. A static feedback controller K, given by

$$u = Ky = \begin{bmatrix} k_1 & k_2 \end{bmatrix} y$$

will be used throughout.

(a) Find a state-space realization for $\mathscr{F}_l(P(s), K)$, i.e. the closed loop transfer function from w to z when K above is used to close the loop from y to u. [20%]

(b) Find the poles of $\mathscr{F}_l(P(s), K)$. Hence show that the closed-loop system is stable if $k_1 < 0$ and $k_2 < 0$. [10%]

(c) Solve

$$A_c^T L + L A_c^T + C_c^T C_c = 0$$

for L, where A_c and C_c denote the "A" and "C" matrices of the closed loop transfer function $\mathscr{F}_l(P(s), K)$. Hence find the $\mathscr{H}_2$ norm of $\mathscr{F}_l(P(s), K)$. [50%]

(d) Find

$$\inf_{k_1, k_2} \|\mathscr{F}_l(P(s), K)\|_2$$

Is there a controller which achieves this infimum? [20%]

END OF PAPER