ENGINEERING TRIPOS PART IIB

Monday 7 May 2007 2.30 to 4

Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Explain what is meant by an *invariant set* in the context of nonlinear dynamical systems, and name *two* kinds of invariant set that can occur. [20%]

(b) State the *circle criterion* for the stability of feedback systems in the form of Fig. 1, in which the nonlinear gain ψ satisfies

$$\alpha \leq \psi(e) \leq \beta \quad \text{for all } e \qquad (1)$$

assuming that $\alpha > 0$. [20%]

(c) Figure 2 shows a feedback system, in which k is a constant gain, and $H(s)$ is the transfer function of a linear system. Show that, for the purposes of stability analysis, this system can be put into the form shown in Fig. 1, with [20%]

$$G(s) = \frac{k + H(s)}{s}$$

(d) With reference to Figs. 1 and 2, inequality (1), and still assuming that $\alpha > 0$, suppose that for a given value k_0 of k, the system satisfies the circle criterion, and

$$\mathrm{Im}\{G(j\omega)\} < 0 \quad \text{for} \quad \omega > 0.$$

Show that the system also satisfies the circle criterion for any larger gain, that is for any $k > k_0$. [30%]

Show that the transfer function $H(s)$ itself is stable. [10%]

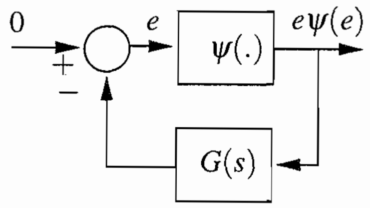

Fig. 1

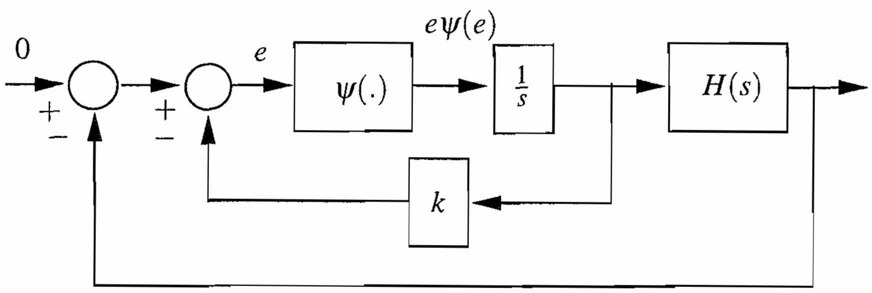

Fig. 2

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2 (a) State *LaSalle's Theorem* and explain why it is often more useful than Lyapunov's Theorems for establishing asymptotic stability. [20%]

(b) The yaw angle ψ of an underwater vehicle responds to an applied torque u according to the differential equation

$$\ddot{\psi} + a\dot{\psi}|\dot{\psi}| = u \qquad (a > 0)$$

Express this equation in state-space form. [10%]

(c) Referring to part (b), if proportional feedback is applied:

$$u = -k\psi$$

show, by considering the function

$$V(\psi, \dot{\psi}) = k\psi^2 + \dot{\psi}^2,$$

that the closed-loop system has an asymptotically stable equilibrium at $(\psi = 0, \dot{\psi} = 0)$ if $k > 0$. [30%]

(d) Referring again to part (b), suppose now that feedback is proportional with saturation:

$$u = -\text{sat}(k\psi) = \begin{cases} +1 & \text{if} \quad k\psi \leq -1 \\ -k\psi & \text{if} \quad -1 < k\psi < 1 \\ -1 & \text{if} \quad k\psi \geq +1 \end{cases}$$

Sketch the form of the state trajectories in the state space, and hence deduce that the closed-loop system is globally asymptotically stable if $k > 0$. [40%]

3 Consider the following nonlinear discrete-time system

$$x(k+1) = f[x(k), u(k)], \qquad \text{with } 0 = f[0,0]$$

and the one-step cost function

$$V(x, u_0) := \ell(x_0, u_0) + F(x_1)$$

where $\ell(\cdot,\cdot)$ and $F(\cdot)$ are given nonlinear functions, $x_0 = x$ is the current, measured value of the system state, and $x_1 := f[x_0, u_0]$.

(a) For a given x, let $u_0^*(x)$ denote the control input that minimises $V(x, u_0)$, and let $V^*(x) := V(x, u_0^*(x))$ be the minimum value. Show that [30%]

$$V(f[x, u_0^*(x)], 0) = V^*(x) - \ell\left(x, u_0^*(x)\right) - F\left(f[x, u_0^*(x)]\right) \\ + \ell\left(f[x, u_0^*(x)], 0\right) + F\left(f\left[f[x, u_0^*(x)], 0\right]\right).$$

(b) Suppose that the function $F(\cdot)$ is known to satisfy

$$F(f[x,0]) - F(x) + \ell(x,0) \leq 0$$

for all x. Using the result of part (a), show that [25%]

$$V^*(f[x, u_0^*(x)]) - V^*(x) \leq -\ell(x, u_0^*(x)).$$

(c) State any additional conditions required on the functions $V^*(\cdot)$ and $\ell(\cdot,\cdot)$ in order to ensure that $V^*(\cdot)$ is a Lyapunov function for the closed loop system [25%]

$$x(k+1) = f[x(k), u_0^*(x(k))].$$

(d) Suppose that the function $F(\cdot)$ satisfies $F(0) = 0$ and $F(x) > 0$ if $x \neq 0$. Recalling the assumptions of part (b) and any additional conditions on the function $\ell(\cdot,\cdot)$ you suggested in (c), what can be inferred about the stability of the uncontrolled system [20%]

$$x(k+1) = f[x(k), 0]?$$

(TURN OVER

4 Consider the following linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

where $x(k)$ is the state and $u(k)$ is the input. Let x_s and u_s be the predicted state and input, respectively, at time $k+s$, i.e. $x_0 = x(k)$ and $x_{s+1} = Ax_s + Bu_s$ for $s = 0, 1, \ldots$. Suppose that the predicted input u_s is modelled as a linear function of the predicted state x_s:

$$u_s = Kx_s + v_s$$

for some matrix K. Define

$$X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad V := \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}.$$

(a) Find matrices Φ and Γ such that [30%]

$$X = \Phi x(k) + \Gamma V.$$

(b) Suppose that the following constraints are given:

$$Cu_s \leq e \text{ for } s = 0, 1; \qquad Dx_2 \leq f.$$

Compute matrices E, F and G and a vector g such that these constraints can be written as [20%]

$$Ex(k) + FX + GV \leq g$$

(c) Using the results of (a) and (b), find matrices S and T and a vector h, in terms of Φ, Γ, E, F, G and g, such that the constraints in (b) can be rewritten as [20%]

$$SV \leq h + Tx(k)$$

(d) Suppose that the following finite-horizon cost function is specified:

$$J(x(k), V) = X^T QX + |c^T V|$$

for a given positive definite matrix Q and a vector c. Show that the problem of finding a vector V that minimizes this cost function subject to the constraints in (c) can be written as a quadratic programming problem. [30%]

Hint: Introduce an additional variable to eliminate the absolute value component of $J(x(k), V)$.

END OF PAPER

Module 4F3: Nonlinear and Predictive Control
Answers 2007

1. —

2. —

3. —

4. (a)

$$\Phi = \begin{bmatrix} A+BK \\ (A+BK)^2 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} B & 0 \\ (A+BK)B & B \end{bmatrix}$$

(b)

$$E = \begin{bmatrix} CK \\ 0 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ CK & 0 \\ 0 & D \end{bmatrix}, \quad G = \begin{bmatrix} C & 0 \\ 0 & C \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} e \\ e \\ f \end{bmatrix}.$$

Note: The solution to this part is not unique, but the idea is to translate the constraints into the given form as simply as possible.

(c)

$$S = F\Gamma + G, \quad T = -(E + F\Phi), \quad h = g$$

(d) —