ENGINEERING TRIPOS PART IIB

Monday 30 April 2007 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 2 - 14 Mar 2007

1 Consider the real-valued signal

$$u(n) = \beta u(n-1) + w(n),$$

where $w(n)$ is an i.i.d. (independent and identically distributed) sequence with zero mean and variance equal to one. The aim is to learn the unknown value β.

(a) Describe, without mathematical detail, the principal methods for adaptive filtering. Compare and contrast their performance and computational load. [25%]

(b) Assume the signal $u(n)$ is stationary and derive $E\{u(n)^2\}$. [15%]

(c) Describe how you would use the least mean-squares (LMS) algorithm to learn β. What is the minimum mean-squared error (MSE) at the optimal solution? [25%]

(d) Describe how you would use the recursive least squares (RLS) algorithm to solve the same problem and show that the RLS algorithm converges to the optimal solution when its parameter $\lambda = 1$. [35%]

2 The inverse of a symmetric positive definite matrix $\mathbf{R}$ can be expressed as

$$\mathbf{R}^{-1} = \mu \sum_{k=0}^{\infty} (\mathbf{I} - \mu \mathbf{R})^k$$

where μ is a small positive constant and $\mathbf{I}$ is the identity matrix.

(a) Using the decomposition $\mathbf{R} = \mathbf{Q}^{\mathrm{T}} \Lambda \mathbf{Q}$ where $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ and Λ is diagonal, find the range of values for μ for which the sum exists. [30%]

(b) Write down the cost function $J(\mathbf{h})$ for the Wiener filtering problem and derive the Steepest Descent method for its solution, where $\mathbf{h}(n)$ is the solution obtained at step n. [20%]

(c) The global solution to a Wiener filtering problem is given by $\mathbf{h}_{\mathrm{opt}} = \mathbf{R}^{-1}\mathbf{p}$. Show that the identity $\mathbf{h}(n) = \mu \sum_{k=0}^{n-1} (\mathbf{I} - \mu \mathbf{R})^k \mathbf{p}$ is equivalent to the Steepest Descent recursion. [20%]

(d) Sketch the contour diagram of $J(\mathbf{h})$ when $\mathbf{R} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$. Make an approximate sketch of the evolution $\mathbf{h}(n)$ of the Steepest Descent recursion on the same contour plot when (i) $\mu = 0.01$ and (ii) $\mu = 4$. [30%]

3 (a) Show that the variance of the periodogram estimate is approximately proportional to the square of the power spectrum for a *Gaussian random process*, i.e. show that

$$\mathrm{var}(\hat{S}_X(e^{j\omega T})) \approx (S_X(e^{j\omega T}))^2$$

where $\hat{S}_X(e^{j\omega T})$ is the periodogram estimate and $S_X(e^{j\omega T})$ is the true power spectrum. [40%]

(b) In a spectrum analyser unit, power spectrum estimates are updated sequentially frame by frame through a long sequence of data, as follows:

$$\hat{P}_X^{(1)}(e^{j\omega T}) = \hat{S}_X^{(1)}(e^{j\omega T})$$
$$\hat{P}_X^{(n)}(e^{j\omega T}) = \alpha\hat{P}_X^{(n-1)}(e^{j\omega T}) + (1-\alpha)\hat{S}_X^{(n)}(e^{j\omega T}), \ n = 2,3,\ldots, \text{ where } 0 < \alpha < 1$$

where $\hat{P}_X^{(n)}(e^{j\omega T})$ is the power spectrum estimtate obtained at frame n in the data and $\hat{S}_X^{(n)}(e^{j\omega T})$ is the periodogram of frame n. The nth frame of data is defined as the set of data points $[x_{(n-1)N}, \ldots, x_{nN-1}]$ where N (the framelength) is the number of data points for the periodogram.

If the periodograms obtained for different frames can be considered statistically independent of one another, show that in the steady state (i.e. as n becomes large) the variance of the estimate $\hat{P}_X^{(n)}(e^{j\omega T})$ is

$$\frac{(1-\alpha)}{(1+\alpha)}(S_X(e^{j\omega T}))^2$$

(you may assume that the result shown in part (a) applies exactly in this case). [30%]

Compare this result with that in part (a) above and explain the effect when α is (i) close to zero and (ii) close to unity. [15%]

Explain how the method could be used for analysis of non-stationary signals whose power spectrum changes slowly and smoothly with time. [15%]

4 A moving average (MA) process of order Q is expressed in the following form:

$$x_n = \sum_{q=0}^{Q} b_q w_{n-q}$$

where $\{w_n\}$ is a zero-mean white noise process having unity variance.

(a) Determine the power spectrum of the MA process and show that the autocorrelation function takes the following form:

$$R_{XX}[r] = \begin{cases} \sum_{q=r}^{Q} b_q b_{q-r} & \text{if } |r| \leq Q \\ 0 & \text{if } r > Q \end{cases}$$

[30%]

(b) The bilateral z-transform for a sequence h_n is defined as

$$\mathscr{Z}\{h_n\} = \sum_{n=-\infty}^{+\infty} h_n z^{-n}$$

Show that the bilateral z-transform of $R_{XX}[r]$ is equal to $B(z)B(z^{-1})$, where

$$B(z) = \sum_{n=0}^{Q} b_n z^{-n}$$

[20%]

(c) If the Q zeros of the polynomial $B(z^{-1})$ are located at n_i, for $i = 1, \ldots, Q$, show that $B(z)$ has zeros at positions $1/n_i$ for $i = 1, \ldots, Q$. [10%]

(d) Use these results to explain the *spectral factorisation* method for estimation of the parameters of a MA model. State clearly any assumptions you make about the model. [20%]

(e) Two values of the autocorrelation function for a MA process are measured as follows:

$$R_{XX}[0] = 5, R_{XX}[1] = 2$$

Determine from this data a suitable MA model of order $Q = 1$. [20%]

END OF PAPER