ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Friday 4 May 2007 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachment: Data Sheet for 4M12* (3 sides).

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

Version 5

1 (a) Without detailed calculations explain briefly, from first principles, how the method of calculus of variations can be used to find the differential equation and boundary conditions satisfied by a function $y(x)$ which minimises the integral

$$\int_a^b F(y, y', x)dx$$

where $F$ is any given function. [20%]

(b) An elastic beam of length $L$, mass per unit length $m$ and bending stiffness $EI$ is pinned at both ends, and restrained against rotation at both ends by identical torsional springs of stiffness $K$ as shown in Fig. 1. The beam can undergo transverse bending vibration, with displacement $y(x,t)$. The potential and kinetic energies associated with this bending motion are

$$V_{beam} = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \quad \text{and} \quad T_{beam} = \frac{m}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx$$

respectively. The potential energy stored in a torsion spring of stiffness $K$ when rotated through angle $\theta$ is $\frac{1}{2}K\theta^2$.


Fig. 1

For a vibration mode, $y(x,t) = u(x)e^{i\omega_n t}$ where $\omega_n$ is the natural frequency. Use Rayleigh's principle and the method of calculus of variations to show that

$$EI\frac{d^4 u}{dx^4} = m\omega_n^2 u .$$

Find the boundary conditions to be satisfied at $x = 0, L$. [40%]

(c) For the case $K = 0$, show that the function

$$u = \sin\left(\frac{n\pi x}{L}\right) \text{ where } n = 1,2,3,...$$

satisfies the equation and the boundary conditions, and deduce the corresponding natural frequency $\omega_n$. [15%]

(d) For the case when $K$ is non-zero but very small, use Rayleigh's principle to obtain an approximate value for the modified natural frequency. [25%]

2 (a) Use suffix notation to calculate (i) $\nabla(1/r)$ and (ii) $\nabla^2(1/r)$

where $r = |\mathbf{x}|$ and $\mathbf{x}$ is the position vector. [35%]

(b) Evaluate the integral

$$I = \iiint_V \nabla^2(1/r)\, dV$$

(i) over a volume $V$ not including the origin $\mathbf{x} = 0$;

(ii) by first transforming to a surface integral, over a volume $V$ which is a sphere of radius $a$ centred on the origin;

(iii) over *any* volume $V$ which includes the origin. [35%]

(c) Comment on the results of (b). To what well-known one-dimensional result does this behaviour correspond? What is the relevance of this result to the solution of Poisson's equation in infinite space? [30%]

3 (a) Explain what is meant by a well-posed problem in the context of second order partial differential equations. [15%]

(b) Show, carefully, that the following problems are *not* well-posed and, for each case, indicate a change to the problem which would make it well-posed.

(i) The function $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial x \partial y} + 2\frac{\partial^2 \phi}{\partial y^2} = 0 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

subject to the boundary conditions

$$\frac{\partial \phi}{\partial y}(x,0) = \frac{\partial \phi}{\partial y}(x,1) = \frac{\partial \phi}{\partial x}(0,y) = \frac{\partial \phi}{\partial x}(1,y) = 0$$ [15%]

(ii) The function $\phi(x, y)$ satisfies

$$\nabla^2 \phi = 0 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

subject to the boundary conditions

$$\frac{\partial \phi}{\partial y}(x,0) = 1\,, \qquad \frac{\partial \phi}{\partial x}(0,y) = \frac{\partial \phi}{\partial x}(1,y) = \frac{\partial \phi}{\partial y}(x,1) = 0$$

and to the auxiliary condition $\phi(0,0) = 0$. [30%]

(iii) The function $\theta(x,t)$ satisfies

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \qquad 0 \le x \le L,\ 0 \le t \le T$$

subject to the boundary conditions

$$\theta(0,t) = \theta(L,t) = 0 \quad \text{and} \quad \theta(x,T) = \sin\frac{\pi x}{L},$$

where $\alpha$ is a positive constant. [40%]

4 (a) Explain briefly what is meant by hyperbolic partial differential equations and the role that characteristic variables play in their solution. [20%]

(b) The function $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2y}\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2y^2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2y^3}\frac{\partial u}{\partial y} = 0 \qquad (1)$$

for $x > 0$ and all $y$, subject to the boundary conditions

$$u(0, y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(0, y) = y^2 .$$

Show that the partial differential equation (1) is of hyperbolic type for $y \neq 0$. [15%]

(c) Show that characteristic variables $(\xi, \eta)$ for equation (1) satisfy

$$x = \xi + \eta \quad \text{and} \quad y^2 = \xi - 2\eta$$

and find the general solution of equation (1) in these variables. [25%]

(d) By solving in characteristic variables equation (1) plus its boundary conditions, show that

$$u(x, y) = \frac{x^2}{2} + xy^2 \qquad [40\%]$$

**END OF PAPER**

## Answers

1 (d) $\omega_{new}^2 \approx \omega_{old}^2 + \dfrac{4Kn^2\pi^2}{mL^3}$

2 (a)(i) $-\dfrac{\mathbf{x}}{|\mathbf{x}|^3}$; (ii) 0 (except at $\mathbf{x} = 0$)

(b)(i) 0; (ii) $-4\pi$; (iii) $-4\pi$