ENGINEERING TRIPOS PART IIB
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Thursday 26 April 2007 9 to 10.30

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

*Answer not more than **three** questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4M13 datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (a) The impulse response function of a dynamic system is given by the integral

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega_n^2 + 2i\beta\omega_n\omega - \omega^2}d\omega$$

where ω_n is the natural frequency of the system and β is the damping ratio ($\beta \ll 1$). By using contour integration, evaluate the impulse response function, carefully distinguishing between the cases $t < 0$ and $t > 0$. [50%]

(b) The following result can be obtained by elementary methods of integration

$$\int_0^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{2}$$

Confirm this result by considering the following contour integral in the complex plane

$$J = \oint_{C_1+C_2+C_3+C_4}\frac{\ln(z)}{1+z^2}dz$$

Here C_1 is a small circle around the origin, C_2 lies just above the positive real axis, C_3 is a circle of infinite radius centred on the origin, and C_4 lies just below the real axis. [50%]

2 (a) The function $f(z) = u + \mathrm{i}v$ is an analytic function of the complex variable $z = x + \mathrm{i}y$. The Cauchy-Riemann equations which relate u and v are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(i) Explain the considerations which lead to the Cauchy-Riemann equations and hence derive these equations. [20%]

(ii) The real part of an analytic function is $u = 2xy$. Deduce the complex part of the function and hence find the complete function $f(z)$. [20%]

(b) Briefly describe what is meant by a Laurent series expansion of a complex function and compare this to a Taylor series expansion. [10%]

(c) A function $f(z)$ has the form

$$f(z) = \frac{g(z)}{(z-a)^3}$$

where $g(z)$ has no singularities.

(i) Expand $g(z)$ about the point $z = a$ using a Taylor series, and hence derive the first four terms in the Laurent series expansion of $f(z)$ about this point. Deduce the value of the residue of $f(z)$ at $z = a$. [20%]

(ii) Evaluate the following integral

$$J = \text{P.V.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{(x-2)^3}\,\mathrm{d}x$$ [30%]

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SECTION B

3 (a) Starting from the Taylor series expansion for the value of a function $f(\mathbf{x})$ at a point $\mathbf{x}_{k+1}$ near a point $\mathbf{x}_k$, derive Newton's Method, i.e. show that successive estimates of the location of the minimum of $f(\mathbf{x})$ are given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \boldsymbol{H}(\mathbf{x}_k)^{-1} \nabla f(\mathbf{x}_k)$$

where ∇f is the gradient of f and $\boldsymbol{H}$ is its Hessian.

Briefly discuss some of the advantages and disadvantages of Newton's Method. [30%]

To minimise the bearing load F of a dual bailer twister drive mechanism an engineer can adjust L, the distance between the traverse bar and the mechanism pivot, and R, the distance from the pivot to the attachment point of the connecting rod.

Analysis shows that

$$F \propto \frac{2}{u}\left(\frac{J}{L^2} + \frac{mL}{3} + M_1\right) + M_2 u$$

where $u = R/L$, J is the mechanism's effective moment of inertia, m is the mass per unit length of the arm connecting the mechanism to the traverse bar, M_1 is the mass of the twister and M_2 is the mass of the connecting rod.

For the design under consideration $J = 2\ \text{kgm}^2$, $m = 3\ \text{kgm}^{-1}$, $M_1 = 8\ \text{kg}$ and $M_2 = 5\ \text{kg}$.

(b) Taking the control variables to be L and u, complete one iteration of Newton's Method from an initial solution $L_1 = 1\ \text{m}$ and $u_1 = 1$. [45%]

(c) Using appropriate optimality criteria find the values of L and u that minimise F, and hence comment on the performance of Newton's Method observed in (b). [25%]

4 (a) Explain with illustrative examples how *slack variables* can be used to convert inequality constraints into equality constraints in linear programming problems. [10%]

A manufacturer produces two products A and B. Both products are made from raw materials C and D. To produce 1 kg of product A requires 0.4 kg of C and 0.6 kg of D. To produce 1 kg of product B requires 0.5 kg of C and 0.5 kg of D. The manufacturer has 100 kg of C and 80 kg of D available. Product A sells at a profit of £12 per kg, while product B sells at a profit of £10 per kg. The manufacturer's agent reports that in the current market he can sell up to 70 kg of product A and 120 kg of product B.

(b) Set up a linear programming problem in standard form to identify how much of each product the manufacturer should produce to maximise profit. Let x_1 and x_2 be the amounts of A and B produced respectively. Include slack variables as required to handle inequality constraints. [20%]

(c) Solve the linear programming problem set up in (b) using phase 2 of the Simplex Method. A suitable initial feasible solution is one in which $x_1 = x_2 = 0$ and slack variables take appropriate values. [70%]

END OF PAPER

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OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*) f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

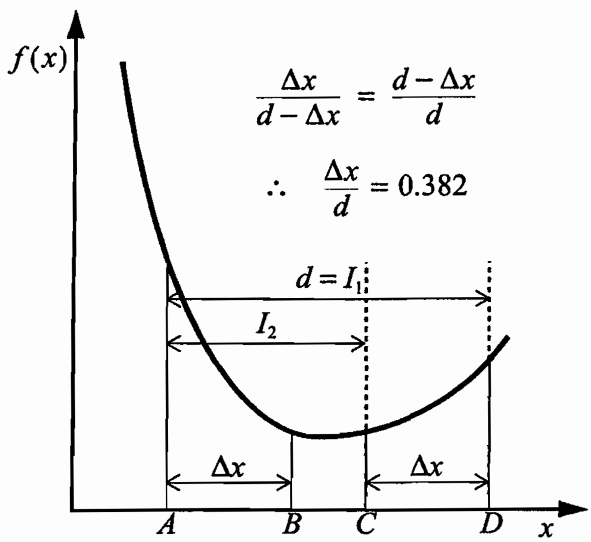
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1}\,J(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } J(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \left[\nabla h_1(\mathbf{x}^*) \ \ldots \ \nabla h_m(\mathbf{x}^*) \right] = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \leq 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall \ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \geq 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$

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Numerical Answers

Q1 (a) For $t > 0$: $x(t) = \frac{1}{\omega_1} e^{-\beta\omega_n t} \sin(\omega_1 t)$

For $t < 0$: $x(t) = 0$

Q2 (a) (ii) $f(z) = -iz^2 + Ci$

(c) (i) $\frac{1}{2} g''(a)$

(ii) $-\frac{1}{2}\pi i e^{2i}$

Q3 (b) $L_2 = 1.159\,\text{m}, U_2 = 1.365$

(c) $L^* = 1.587\,\text{m}, U^* = 2.038$

Q4 (b) Minimise $f(\bar{x}) = -12x_1 - 10x_2$

Subject to
$$0.4x_1 + 0.5x_2 + x_3 = 100$$
$$0.6x_1 + 0.5x_2 + x_4 = 80$$
$$x_1 + x_5 = 70$$
$$x_2 + x_6 = 120$$

(c) $x_1^* = 70\,\text{kg}, x_2^* = 76\,\text{kg}$