Dr M.P. Juniper

$p'(a^+, x, t) = p_0 e^{i(\omega t + kx)}$

Qu1

a) The liquid motion in $r<a$ is incompressible and inviscid and initially at rest. Hence it is irrotational and the velocity potential satisfies $\nabla^2\phi = 0$. In order to match the wall pressure, write

$$\phi(r,x,t) = f(r)\, e^{i(\omega t + kx)} \qquad (1)$$

In cylindrically polar co-ordinates $\nabla^2\phi = 0$ is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

(Maths data book)

Substituting for ϕ from (1) this reduces to

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - k^2 f = 0.$$

The 4A10 Data card states that this is the modified Bessel equation with independent solutions $I_0(kr)$ and $K_0(kr)$. $K_0(kr)$ is infinite on $r=0$ but we require a finite fluid velocity there. Hence $f(r) = A\, I_0(kr)$ where A is a constant and

$$\underline{\phi(r,x,t) = A\, I_0(kr)\, e^{i(\omega t + kx)}}$$

[20%]

b) The pressure perturbation in the core is given by

$$p(r,x,t) = -\rho_1 \frac{\partial\phi}{\partial t} = -\rho_1\, i\omega A\, I_0(kr)\, e^{i(\omega t + kx)}$$

Hence on the inner surface of the hose $p = -\rho_1 i\omega A I_0(ka) e^{i(\omega t + kx)}$

The radial displacement can be determined from $\dfrac{\partial\eta}{\partial t} = \dfrac{\partial\phi}{\partial r}$.

Hence the hose radial displacement is given by

$$\eta = \frac{1}{i\omega} A k\, I_0'(ka)\, e^{i(\omega t + kx)}.$$

Substituting into the equation for the hose:

$$p_0 e^{i(\omega t + kx)} + \rho_1 i\omega A\, I_0(ka) e^{i(\omega t + kx)} = -\frac{Ed}{a^2}(1+i\delta)\frac{1}{i\omega} A k\, I_0'(ka) e^{i(\omega t + kx)}$$

Hence $A\left[\rho_1 \omega^2 I_0(ka) - \dfrac{Ed}{a^2}(1+i\delta)\, k I_0'(ka)\right] = i\omega p_0$

Qu1 cont.)

$$A = \frac{i\omega p_0}{\rho_1 \omega^2 I_0(ka) - \frac{Ed}{a^2}(1+i\delta) k I_0'(ka)}$$

The pressure on the centre-line $= -\rho_1 i\omega A I_0(0) e^{i(\omega t + kx)}$

$$= \frac{\rho_1 \omega^2}{\rho_1 \omega^2 I_0(ka) - \frac{Ed}{a^2}(1+i\delta) k I_0'(ka)} p_0 e^{i(\omega t + kx)}$$

using $I_0(0) = 1$ [50%]

c) For small ka, $I_0(ka) = 1$

$I_0'(ka) = \frac{1}{2} ka$

Hence centre-line pressure $= \dfrac{\rho_1 \omega^2}{\rho_1 \omega^2 - \frac{Ed}{a^2}(1+i\delta)\frac{1}{2}k^2 a} p_0 e^{i(\omega t + kx)}$

$$= \frac{\omega^2}{\omega^2 - c_b^2(1+i\delta)k^2} p_0 e^{i(\omega t + kx)}$$

where $c_b = \left(\dfrac{Ed}{2\rho_1 a}\right)^{1/2}$

(i) when $\omega \gg c_b k$, centre-line pressure $= p_0 e^{i(\omega t + kx)}$ = wall pressure.

The long wavelength acoustic waves are detected by the centre-line hydrophones unattenuated,

(ii) when $\omega \ll c_b k$ centre-line pressure $\approx \dfrac{-\omega^2}{c_b^2 k^2 (1+i\delta)} p_0 e^{i\omega(t+kx)}$

centre-line pressure is a factor of order $(\omega / c_b k)^2$ smaller than the wall pressure. The short-wavelength pressure fluctuations are significantly attenuated. Typical short-wavelength disturbances are convected turbulent fluctuations, wavelength $\sim 2\pi \bar{U}/\omega$.

(iii) when $\omega \approx c_b k$ the centre-line pressure is very large compared to the wall pressure. There are large radial hose displacements and internal pressures with this phase speed. These are 'bulge' or varicose waves travelling along the cylinder. The response to forcing at the bulge wave speed c_b is finite only because of damping in the hose. [30%]

APD

Qu2 a) Bookwork - see section 3.1 in Lecture Notes [30%]

b) Viscosity has a stabilising influence. Flows that are predicted to be unstable according to inviscid theory will be stable below a critical Reynolds number. [5%]

c)(i) One example would be a shaft rotating in an outer casing, the gap lubricated by oil. Here the onset of the instability would be disadventageous because the torque due to drag on the rotating shaft would increase.

(ii) Since the onset of the instability leads to vortices with their axes in the direction of the main flow, mixing and heat transfer from the wall to the fluid is enhanced, and there are lots of examples where that might be advantageous. [5%]

d) Rayleigh's criterion is that the flow is inviscidly unstable if Γ^2 decreases with increasing radius r, where $\Gamma = 2\pi V r$ is the circulation.

(i) In this flow, the fluid is in rigid body motion with Ω uniform

$$V = \Omega r$$

$$\Gamma = 2\pi \Omega r^2$$ which always increases with increasing r. Hence the flow is <u>stable</u> [15%]

(ii) In this flow $V = \dfrac{k}{r}$ which is the form of the viscous flow with circular streamlines that tends to zero as $r \to \infty$ (the other term is Ar)

$\Gamma = 2\pi k$ is constant with r and this flow is <u>neutrally stable</u> [15%]

(iii)
$$\Omega = 0 \qquad r \leq R_1 \qquad \Rightarrow \Gamma = 0$$

$$\Omega = A + \frac{B}{r^2} \quad \text{for } R_1 \leq r \leq R_2 \quad \Rightarrow \Gamma = (Ar^2 + B)2\pi$$

$$= \Omega_0 \frac{R_2^2}{r^2} \quad \text{for } R_2 \leq r \quad \Rightarrow \Gamma = (\Omega_0 R_2^2) 2\pi$$

Continuity at $r = R_1$ $\quad AR_1^2 + B = 0$

Continuity at $r = R_2$ $\quad AR_2^2 + B = \Omega_0 R_2^2$

Subtracting $\quad A(R_2^2 - R_1^2) = \Omega_0 R_2^2 \Rightarrow A = \dfrac{\Omega_0 R_2^2}{R_2^2 - R_1^2}$

[30%]

Qu 2 cont.) $B = -AR_1^2 = -\dfrac{\Omega_0 R_2^2 R_1^2}{R_2^2 - R_1^2}$

Hence in $R_1 \leq r \leq R_2$ $\Gamma = \dfrac{\Omega_0 R_2^2 (r^2 - R_1^2)}{R_2^2 - R_1^2}$

Hence Γ always has the same sign as Ω_0.

$\Gamma^2 =$ constant in $r < R_1$ $\longrightarrow$ neutrally stable

Γ^2 increases with r in $R_1 \leq r \leq R_2$ $\longrightarrow$ stable

Γ^2 constant in $r > R_2$ $\longrightarrow$ neutrally stable

Overall this flow is <u>neutrally stable</u>.

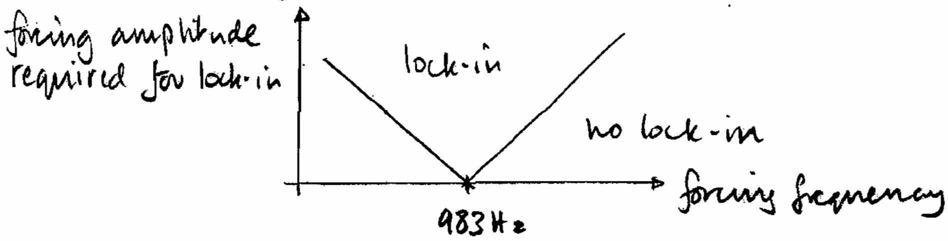
APD

3 a) Helium jet

i) When unforced, the PSD of the velocity signal shows a clear peak at 983 Hz. When strongly forced (e.g. at 900 mVrms) there is an equally clear peak at the forcing frequency (1023 Hz). This clear peak is visible once the forcing reaches 700 mVrms, at which point the jet's response has locked into the forcing frequency. [20%]

At intermediate forcing, the natural frequency at 983 Hz and the forcing frequency at 1023 Hz co-exist. The flow behaves like a coupled oscillator [10%]

ii) Qualitatively the same features would be seen for forcing at 1000 Hz and at 1050 Hz. The forcing signal that is required to achieve lock in, however, would be different. For 1000 Hz, which is closer to the jet's natural frequency, a lower forcing amplitude would be required. For 1050 Hz, which is further from the jet's natural frequency, a higher forcing amplitude would be required. In general, the plot of lock-in amplitude vs forcing frequency looks like this:



[10%]

b) Under-sea walkway.

The cross-section is circular so there will be no gallop or flutter. The walkway will, however, potentially suffer from vortex shedding. This will be particularly dangerous if the frequency of vortex shedding locks into the resonant frequency of the walkway. [15%]

To calculate the walkway's resonant frequency in water, f_w, we can use the resonant frequency in air, f_a, but must consider the added mass of the water around the structure. The stiffness will be unchanged and, since $f \propto \sqrt{k/m}$,

$$f_w = f_a \sqrt{\frac{m + m_a}{m + m_w}}$$ [20%]

where m is the mass per unit length of the structure, m_a is the added mass in air and m_w is the added mass in water.

The added mass coefficient for a cylinder is 1 so the added mass is equal to $1 \times \rho \pi D^2/4$. Therefore

$$f_W = 0.5 \sqrt{\frac{885 + 1.2\pi\, 3^2/4}{885 + 1027\pi\, 3^2/4}} = 0.166 \text{ Hz}$$

The pipe's diameter is much smaller than the water's depth and it is neither near the surface or the sea floor. Therefore we do not need to consider the effects of the floor or the surface on the added mass coefficient. [5%]

The Reynolds number is much greater than 1000 so the Strouhal number of vortex shedding is 0.2. The frequency of vortex shedding = $St\, U/D$ and this is equal to 0.166 Hz when $U = 2.49\ ms^{-1}$. The maximum cross-stream velocity is $3 ms^{-1}$ so lock-in can potentially occur. This would be dangerous. [20%]

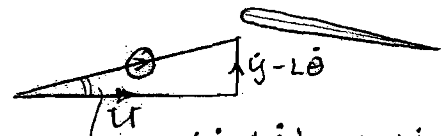
MPJ

4.

(a) Front aerofoil:

angle of attack if stationary $= \theta$
vertical velocity (positive downwards) $= \dot{y} - L\dot{\theta}$

In a frame of reference moving with the aerofoil the incidence angle changes



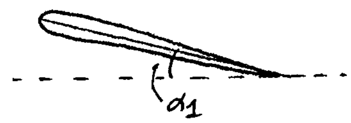
$\tan^{-1}\left(\dfrac{\dot{y} - L\dot{\theta}}{U}\right) \simeq \dfrac{\dot{y} - L\dot{\theta}}{U}$ for small angles

$\Rightarrow$ apparent angle of attack $= \theta + \dfrac{\dot{y} - L\dot{\theta}}{U} = \alpha_1$

Rear aerofoil: vertical velocity (positive downwards) $= \dot{y} + L\dot{\theta}$

$\Rightarrow$ apparent angle of attack $= \theta + \dfrac{\dot{y} + L\dot{\theta}}{U} = \alpha_2$ [20%]

(b) Front aerofoil:



α_1 is defined positive in the direction shown (i.e. clockwise)

Force upwards on aerofoil $= \frac{1}{2}\rho U^2 A_1 \left(\underbrace{\dfrac{\partial C_L}{\partial \alpha}\alpha_1}_{\text{Lift coefficient at angle } \alpha_1}\right)$

n.b. F_1 is defined positive downwards, hence minus sign below:

so $F_1 = -\frac{1}{2}\rho U^2 A_1 \dfrac{\partial C_L}{\partial \alpha}\left(\theta + \underbrace{\dfrac{\dot{y} - L\dot{\theta}}{U}}\right)$

we are told to neglect these terms

Rear aerofoil: $F_2 = -\frac{1}{2}\rho U^2 A_2 \dfrac{\partial C_L}{\partial \alpha}\left(\theta + \overbrace{\dfrac{\dot{y} + L\dot{\theta}}{U}}\right)$ [10%]

n.b. the sign convention would be simplified by defining y positive upwards but this question follows the convention in the notes and in Blevins' book. Students would not be penalized for re-defining the sign convention.

(c) $k_y = 2k$, $k_\theta = \dfrac{kL^2}{2}$, $F_y = F_1 + F_2$, $F_\theta = (F_2 - F_1)L$

$$m\ddot{y} + S_x\ddot{\theta} + k_y y = F_y = F_1 + F_2 = -\tfrac{1}{2}\rho U^2 (A_1 + A_2)\left(\partial C_L/\partial\alpha\,\theta\right) \qquad (1a)$$
$$I_\theta\ddot{\theta} + S_x\ddot{y} + k_\theta\theta = F_\theta = (F_2 - F_1)L = -\tfrac{1}{2}\rho U^2 (A_2 - A_1)\left(\partial C_L/\partial\alpha\,\theta\right)L \qquad (1b)$$

Define constant $\gamma \equiv \tfrac{1}{2}\rho U^2 \dfrac{\partial C_L}{\partial\alpha}$ (this is always positive)

Assume that the torsional mode is coupled with the translational mode and that both can be described by the same complex growth rate, s:

$\Rightarrow y = y_0 e^{st}$ and $\theta = \theta_0 e^{st}$

$\Rightarrow \dot{y} = sy$, $\ddot{y} = s^2 y$, $\dot{\theta} = s\theta$, $\ddot{\theta} = s^2\theta$ [10%]

Re-arrange 1a and 1b into homogenous ODEs:

$$\Rightarrow \quad m\ddot{y} + S_x\ddot{\theta} + k_y y + \gamma(A_1 + A_2)\theta = 0 \qquad (2a)$$
$$\Rightarrow \quad I_\theta\ddot{\theta} + S_x\ddot{y} + k_\theta\theta + \gamma(A_2 - A_1)\theta L = 0 \qquad (2b)$$
[10%]

Substitute in expressions for $\ddot{y}, \dot{y}, y, \ddot{\theta}, \dot{\theta}, \theta$ and divide by e^{st}

$$\Rightarrow \quad ms^2 y_0 + S_x s^2\theta_0 + k_y y_0 + \gamma(A_1 + A_2)\theta_0 = 0 \qquad (3a)$$
$$\Rightarrow \quad I_\theta s^2\theta_0 + S_x s^2 y_0 + k_\theta\theta_0 + \gamma(A_2 - A_1)\theta_0 L = 0 \qquad (3b)$$

Convert to matrix form

$$\begin{bmatrix} ms^2 + k_y & S_x s^2 + \gamma(A_1 + A_2) \\ S_x s^2 & I_\theta s^2 + k_\theta + \gamma(A_2 - A_1)L \end{bmatrix}\begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix} = 0 \qquad (4)$$
[10%]

This only has non-trivial solutions when the determinant of the matrix is zero, which is when the following expression is satisfied:

$$s^4\underbrace{\left[mI_\theta - S_x^2\right]}_{a} + s^2\underbrace{\left[mk_\theta + m\gamma(A_2 - A_1)L + k_y I_\theta - S_x\gamma(A_2 + A_1)\right]}_{b} + \underbrace{k_y k_\theta + k_y\gamma(A_2 - A_1)L}_{c} \quad ($$

Solve this quadratic equation for s^2: $\quad s^2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad ($

[10%]

The system is unstable if s has a positive real component, i.e. if s^2 is complex or a positive real number, then the system is unstable. The system is only stable if s^2 is negative real.

i) s^2 will be complex if $4ac > b^2$. This will give rise to complex s, which corresponds to an exponentially growing harmonic motion (there will also be a corresponding decaying motion). This is coupled mode flutter. $4ac > b^2$ can be written as:

Car is unstable to flutter if...

$$4\underbrace{\left[mI_\theta - S_x^2\right]}_{\text{We are told that this is always positive}}\underbrace{\left[k_y k_\theta + k_y \gamma (A_2 - A_1)L\right]}_{\text{This is positive if the car is not unstable to torsional divergence (see (ii))}} > \left[mk_\theta + m\gamma(A_2-A_1)L + k_y I_\theta - S_x\gamma(A_2+A_1)\right]^2 \quad (7)$$

the R.H.S. is always positive

[5%]

ii) One root of s^2 will be a positive real number if $4ac < 0$ because $\sqrt{b^2-4ac}$ will be greater then b and the ~~numerator~~ of equation (6) will be positive and the denominator, we are told, is positive. This reduces to $c < 0$ because a is always positive, which implies:

$$k_y k_\theta + k_y \gamma (A_2 - A_1)L < 0$$

$$\Rightarrow \quad k_\theta < \tfrac{1}{2}\rho U^2 \frac{\partial C_L}{\partial \alpha} L (A_1 - A_2) \quad \text{for instability}$$

n.b. $\frac{\partial C_L}{\partial \alpha}$ is positive

[5%]

This corresponds to torsional divergence – i.e. exponential growth of y and θ with no oscillation (the car flipping over backwards).

Subst. for $k_\theta \Rightarrow \quad \frac{kL^2}{2} < \tfrac{1}{2}\rho U^2 \frac{\partial C_L}{\partial \alpha}(A_1 - A_2)L \Rightarrow kL < \rho U^2 \frac{\partial C_L}{\partial \alpha}(A_1 - A_2)$

(d) torsional divergence: If $A_1 > A_2$ (the front aerofoil is larger than the back aerofoil) then the car will always be unstable to torsional divergence at sufficiently high speed. Its susceptibility to torsional divergence can be reduced by increasing k, increasing L or decreasing $\partial C_L / \partial \alpha$. It can be eliminated by setting $A_2 > A_1$.

coupled mode flutter: There is lots of scope for discussion here from analysis of (7). Many terms appear on both sides of (7), however, so it is difficult to isolate the effect of each parameter. [10%]

The suspension dampers will make the car more stable. However, this damping can always be exceeded at sufficiently high velocity U. [10%]

MPJ