2008 IIB 4A12 TURBULENCE

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1 (a) Helmholtz's 1st Law:

Fluid elements that lie on a vortex line at $t=0$ stay on the line

Helmholtz's 2nd Law:

Flux of vorticity along a vortex tube, $\Phi = \int \underset{\sim}{\omega} \, d\underset{\sim}{A}$, is constant along the tube and independent of time.

Both laws require the fluid to be inviscid.

Vorticity equation for inviscid fluid is

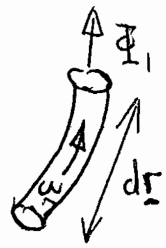
$$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u}$$

For "die line" we have

$$\frac{D}{Dt}(d\underset{\sim}{l}) = (d\underset{\sim}{l}\cdot\nabla)\underset{\sim}{u}$$

Comparing the two we see that $\underset{\sim}{\omega}$ and $d\underset{\sim}{l}$ evolve in the same way, so if they start coincident they must stay coincident.

(b) $$H_1 = \int_{V_1} \underset{\sim}{u}\cdot\underset{\sim}{\omega}\, dV$$



But $\underset{\sim}{\omega}\, dV = |\underset{\sim}{\omega}| A \, d\underset{\sim}{r}$

$= \Phi_1 \, d\underset{\sim}{r}$

$$\Rightarrow H_1 = \oint_{C_1} \underset{\sim}{u}\cdot(\Phi_1 \, d\underset{\sim}{r}) = \Phi_1 \oint_{C_1} \underset{\sim}{u}\cdot d\underset{\sim}{r}$$ (since Φ_1 constant along tube)

case (i)

(unlinked)

Stokes: $\oint_{c_1} \underset{\frown}{u} \cdot d\underset{\frown}{r} = \int_{S_1} \underset{\frown}{\omega} \cdot d\underset{\frown}{s} = 0$

$\Rightarrow \quad H_1 = 0$

similarly $H_2 = 0 \quad \Rightarrow \quad \underline{\underline{H = H_1 + H_2 = 0}}$

case (ii)

(right-handed linkage)

Stokes: $\oint_{c} \underset{\frown}{u} \cdot d\underset{\frown}{r} = \int_{S_1} \underset{\frown}{\omega} \cdot d\underset{\frown}{s} = \Phi_2$

$\Rightarrow \quad H_1 = \Phi_1 \oint_{c_1} \underset{\frown}{u} \cdot d\underset{\frown}{r} = \Phi_1 \Phi_2$

Similarly $H_2 = \Phi_2 \oint_{c_2} \underset{\frown}{u} \cdot d\underset{\frown}{r} = \Phi_2 \Phi_1$

Thus $\quad \underline{\underline{H = H_1 + H_2 = 2\Phi_1 \Phi_2}}$

case (iii)

(left-handed linkage)

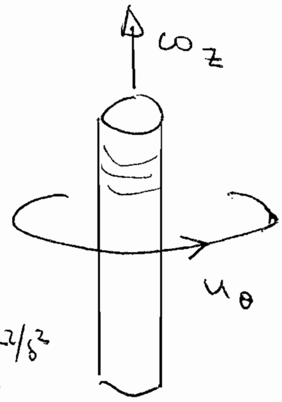
Stokes: $\oint_{c_1} \underset{\frown}{u} \cdot d\underset{\frown}{r} = \int_{S_1} \underset{\frown}{\omega} \cdot d\underset{\frown}{s} = -\Phi_2$

$\Rightarrow \quad H_1 = \Phi_1 \oint_{c_1} \underset{\frown}{u} \cdot d\underset{\frown}{r} = -\Phi_1 \Phi_2$

Similarly $H_2 = \Phi_2 \oint_{c_2} \underset{\frown}{u} \cdot d\underset{\frown}{r} = -\Phi_2 \Phi_1$

$\Rightarrow \quad \underline{\underline{H = H_1 + H_2 = -2\Phi_1 \Phi_2}}$

(c) In an inviscid flow the vortex tubes are locked into the fluid like dye-lines, and so they conserve their topology for all time. Since H is a measure of the topological linkage of the tubes, it is also conserved.



2 (a)
$$\begin{cases} \omega_z = \dfrac{\Gamma_0}{\pi \delta^2} \exp\left[-r^2/\delta^2\right] \\[2ex] u_\theta = \dfrac{\Gamma_0}{2\pi r}\left[1 - \exp(-r^2/\delta^2)\right] \end{cases}$$

- $$\frac{\partial \omega}{\partial t} = \frac{\Gamma_0}{\pi}\left(-2\delta^{-3}\frac{d\delta}{dt}\right) e^{-r^2/\delta^2} + \frac{\Gamma_0}{\pi\delta^2}\left(\frac{2r^2}{\delta^3}\frac{d\delta}{dt}\right) e^{-r^2/\delta^2}$$

$$= \frac{2\Gamma_0}{\pi\delta^3}\frac{d\delta}{dt} e^{-r^2/\delta^2}\left(\frac{r^2}{\delta^2} - 1\right)$$

- $$(\underline{u}\cdot\nabla)\,\omega = u_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\omega = 0$$

- $$\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right) = \frac{\nu}{r}\frac{\partial}{\partial r} r\left[\frac{\Gamma_0}{\pi\delta^2}\left(\frac{-2r}{\delta^2}\right)e^{-r^2/\delta^2}\right]$$

$$= -\frac{\nu}{r}\frac{2\Gamma_0}{\pi\delta^4}\frac{\partial}{\partial r}\left[r^2 e^{-r^2/\delta^2}\right]$$

$$= -\frac{\nu}{r}\frac{2\Gamma_0}{\pi\delta^5}\left[2r e^{-r^2/\delta^2} - \frac{2r^3}{\delta^2}e^{-r^2/\delta^2}\right]$$

$$= \frac{4\nu\Gamma_0}{\pi\delta^4}\left[\left(\frac{r}{\delta}\right)^2 - 1\right]e^{-r^2/\delta^2}$$

equate terms,

$$\frac{2\Gamma_0}{\pi\delta^3}e^{-r^2/\delta^2}\left(\left(\frac{r}{\delta}\right)^2 - 1\right)\frac{d\delta}{dt} = \frac{4\nu\Gamma_0}{\pi\delta^4}\left(\left(\frac{r}{\delta}\right)^2 - 1\right)e^{-r^2/\delta^2}$$

$$\Rightarrow \frac{d\delta}{dt} = \frac{2\nu}{\delta}$$

$$\Rightarrow \frac{d\delta^2}{dt} = 4\nu$$

$$\Rightarrow \delta^2 = 4\nu t$$

$$\Rightarrow \underline{\underline{\delta = \sqrt{4\nu t}}} \qquad (c = 4\nu)$$

(b) The vortex tube thickens by diffusion,

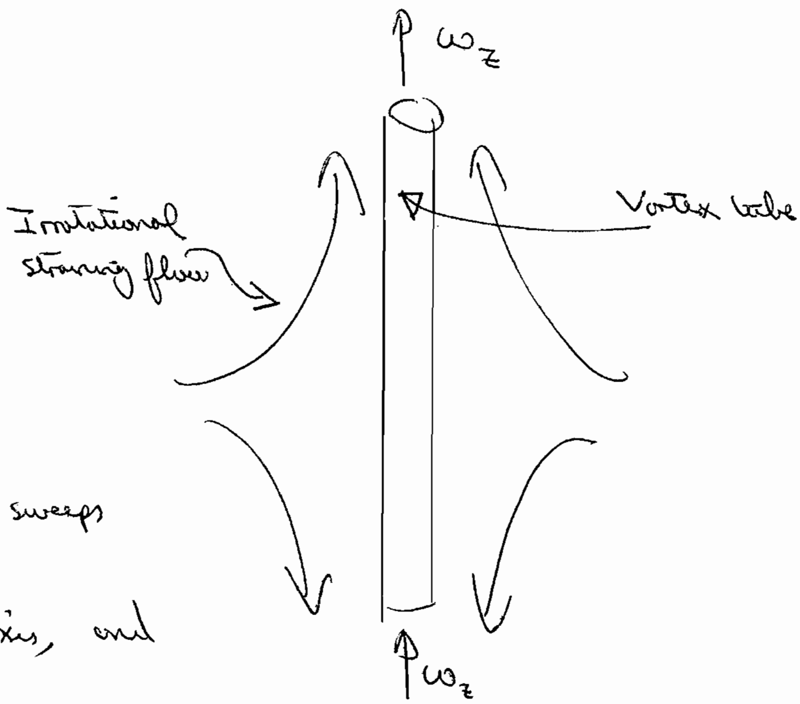
c.f. $\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega$.

The flux along the tube is constant, however,

$$\Phi = \Gamma_0 = \text{constant}. \quad \text{(Cannot create or destroy vorticity in a 2D flow)}$$

Thus the average (and peak) vorticity falls as δ increases, in such a way that Φ is conserved.

(c) Burger's vortex:



In Burger's vortex the irrotational straining flow sweeps vorticity in towards the axis, and this exactly counters the tendency for vorticity to spread outward by diffusion

Also there is an axial strain set up on the axis which tends to intensify the vorticity by vortex-line stretching. This exactly counteracts the tendency of the peak vorticity to fall by diffusion.

3. (a) $$Re_t = \frac{u\, L_{turb}}{\nu} \quad (1)$$

$$\varepsilon \approx \frac{u^3}{L_{turb}} \quad (2)$$

$$\varepsilon = 15\, \nu \frac{u^2}{\lambda^2} \quad (3)$$

$$Re_\lambda \equiv \frac{u\,\lambda}{\nu} = \frac{\lambda}{L_{turb}} \frac{u\,L_{turb}}{\nu} = \frac{\lambda}{L_{turb}} \cdot Re_t$$

From (2) & (3) : $$15\,\nu \frac{u^2}{\lambda^2} = \frac{u^3}{L_{turb}} \Leftrightarrow \frac{\lambda^2}{L_{turb}^2} = 15 \frac{\nu}{u\,L_{turb}}$$

$$\Leftrightarrow \frac{\lambda}{L_{turb}} = \sqrt{15}\; Re_t^{-1/2}$$

$$\Rightarrow Re_\lambda = \sqrt{15}\; Re_t^{-1/2} \cdot Re_t$$

$$\Rightarrow \boxed{Re_\lambda = \sqrt{15}\; Re_t^{1/2}}$$

$$\eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \left(\frac{\nu^3}{u^3} L_{turb}\right)^{1/4} = \left(\frac{\nu^3}{u^3 L_{turb}^3} \cdot L_{turb}^4\right)^{1/4}$$

$$\Leftrightarrow \eta_K = L_{turb}\; Re_t^{-3/4}$$

$$\eta_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \left(\frac{\nu^3}{15\,\nu\, u^2} \lambda^2\right)^{1/4} = 15^{-1/4} \left(\frac{\nu^2}{u^2 \lambda^2}\, \lambda^4\right)^{1/4}$$

$$\Leftrightarrow \eta_K = 15^{-1/4}\, \lambda \cdot Re_\lambda^{-1/2}$$

(b)

Mean streamwise momentum:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \cancel{\nu\frac{\partial^2 \bar{u}}{\partial x^2}} - \frac{\partial \overline{u'v'}}{\partial y} + \cancel{\nu\frac{\partial^2 \bar{u}}{\partial y^2}} - \frac{\partial \overline{u'^2}}{\partial x}$$

Using $-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \frac{\partial \overline{v'v'}}{\partial x}$, x-momentum equation is

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x}}_{\mathrm{I}} + \underbrace{\bar{v}\frac{\partial \bar{u}}{\partial y}}_{\mathrm{II}} = \underbrace{-\frac{\partial \overline{u'^2}}{\partial x}}_{\mathrm{III}} + \underbrace{\frac{\partial \overline{v'^2}}{\partial x}}_{\mathrm{IV}} - \underbrace{\frac{\partial \overline{u'v'}}{\partial y}}_{\mathrm{V}}$$

From continuity $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$, the order of magnitude of $\bar{v} = o\left(\frac{U_c \delta}{L}\right)$

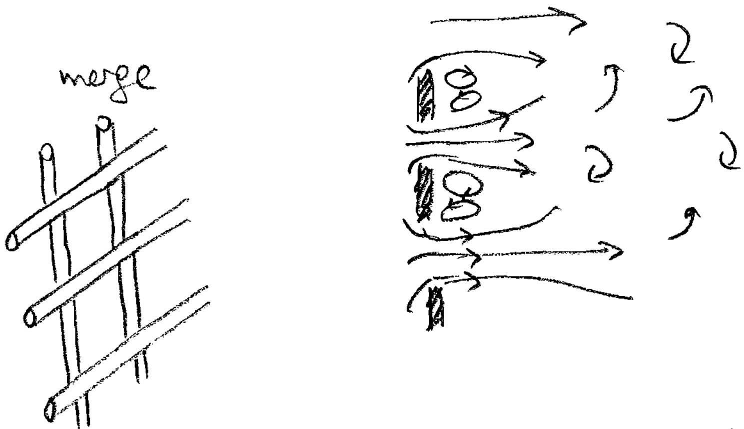
$\Rightarrow$ order of magnitude of terms in x-mom:

$\mathrm{I}: U_c \frac{U_c}{L}$; $\mathrm{II}: \frac{U_c \delta}{L}\frac{U_c}{\delta}$, $\mathrm{III}: \frac{u_0^2}{L}$

$\mathrm{IV}: \frac{u_0^2}{L}$ $\quad$ $\mathrm{V}: \frac{u_0^2}{\delta}$

III & IV are negligible $\Rightarrow \frac{u_0^2}{U_c^2} \sim \frac{\delta}{L}$

4. (a) In wind tunnel turbulence, the velocity fluctuations are created initially when jets and wakes immediately downstream of the grid break down and merge



Immediately downstream of the grid, there is shear ⇒ turbulence not isotropic, and the velocity profile very inhomogeneous ⇒ turbulence is inhomogeneous

Very far from the grid, the pressure correlation terms in the Reynolds stress equation cause a more isotropic field and turbulent transport homogenizes the turbulence in space.

(b) In this problem, the turbulence kinetic energy eqtn becomes

$$\frac{dk}{dt} = -\varepsilon$$

If $k = k_0 \left(\frac{t}{t_0}\right)^{-m}$, $\frac{dk}{dt} = -\left(\frac{m k_0}{t_0}\right)\left(\frac{t}{t_0}\right)^{-(m+1)}$

$$\Rightarrow \quad \varepsilon = \varepsilon_0 \left(\frac{t}{t_0}\right)^{-(m+1)} \qquad \left(\varepsilon_0 = \frac{m k_0}{t_0}\right)$$

The turbulent timescale T is $\frac{k}{\varepsilon}$

$$\Rightarrow T = \frac{k_0}{\varepsilon_0}\left(\frac{t}{t_0}\right)^{-m+(m+1)} \Leftrightarrow T = \frac{k_0}{\varepsilon_0}\left(\frac{t}{t_0}\right)$$

$\frac{k_0}{\varepsilon_0}$ is the initial eddy turn-over time T_0

$$\Rightarrow \quad \underline{\underline{T = T_0 \left(\frac{t}{t_0}\right)}}$$

The turbulent lengthscale L is $\sqrt{k} \cdot T$

$$\Rightarrow \quad L = k_0^{1/2}\left(\frac{t}{t_0}\right)^{-m/2} \cdot T_0 \left(\frac{t}{t_0}\right)$$

$$\Rightarrow \quad \underline{\underline{L = L_0 \left(\frac{t}{t_0}\right)^{1-m/2}}}$$

For $m = 1.2 \rightarrow$ $T = T_0\left(\frac{t}{t_0}\right)$

$L = L_0 \left(\frac{t}{t_0}\right)^{0.4}$

(c) $Re_t = \frac{\sqrt{k}\, L}{\nu} = \frac{\sqrt{k_0}\, L_0}{\nu}\left(\frac{t}{t_0}\right)^{-m} \cdot \left(\frac{t}{t_0}\right)^{1-m/2} = Re_{t,0} \cdot \left(\frac{t}{t_0}\right)^{1-m}$

$$\Rightarrow \quad 1 = 100 \cdot \left(\frac{t}{t_0}\right)^{1-m} \Rightarrow \frac{t}{t_0} = \frac{1}{1-m} \ln(0.01)$$

$$\underline{\underline{\approx 23.}}$$