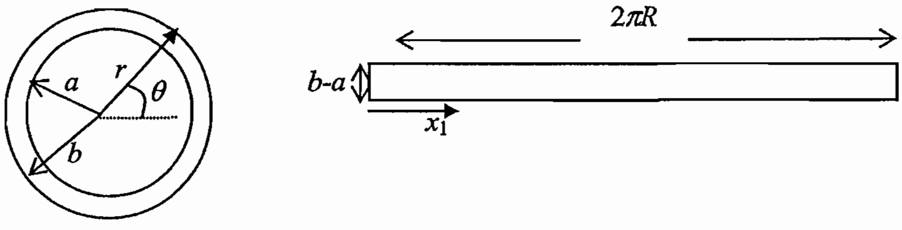
2008 IIB 4A14 SILENT AIRCRAFT INITIATIVE PROF A P DOWLING

Solutions

PROF A P DOWLING

Qu 1

a) When the annulus height $(b-a)$ is small compared with the radius a, the duct curvature is not important and the annulus can be unwound as a rectangular duct of height $h = b-a$ and length $2\pi R$, where R is the average radius $(a+b)/2$. [10%]



annular cross- section of duct

rectangular approximation

b) There are rigid wall boundary conditions on the surfaces $x_2 = 0$ and h which correspond to the inner and outer radii of the annulus;

$$\frac{\partial p'}{\partial x_2} = 0 \quad \text{on} \quad x_2 = 0 \text{ and } h.$$

The requirement that in the annular duct $p'(r,\theta,x_3 t) = p'(r,\theta+2\pi,x_3,t)$ leads to $p'(x_1,x_2,x_3,t) = p'(x_1+2\pi R, x_2, x_3, t)$ in the rectangular duct. [10%]

c) Writing the solution in the form

$$p'(x_1,x_2,x_3,t) = e^{i\omega t} f(x_1) g(x_2) h(x_3)$$

and substituting into the wave equation leads to:

$$\frac{\ddot{g}}{g} + \frac{\ddot{h}}{h} + \frac{\omega^2}{c_0^2} = -\frac{\ddot{f}}{f} = -k^2 \text{ say}$$

$$f = A\cos(kx_1) + B\sin(kx_1)$$

The periodic boundary condition $f(x_1+2\pi R) = f(x_1)$ requires that

$$k = \frac{n}{R} \quad \text{for integer } n.$$

With $\frac{\ddot{g}}{g} = k_2^2$ giving solutions $g(x_2) = C\cos(k_2 x_2) + D\sin(k_2 x_2)$

The rigid wall boundary condition $\frac{dg}{dx_2} = 0$ on $x_2 = 0$ leads to $D = 0$, $k_2 = m\pi/(b-a)$ for m integer.

Module 4A14 2008

Solutions

Qu 1 cont.)

The equation for $h(x_3)$ is therefore

$$\frac{\ddot{h}}{h} = -\frac{\omega^2}{c_0^2} - \frac{\ddot{f}}{f} - \frac{\ddot{g}}{h} = -\frac{\omega^2}{c_0^2} + \left(\frac{n}{R}\right)^2 + \left(\frac{m\pi}{b-a}\right)^2 = -k_{mn}^2 \text{ say}$$

The x_3-dependence is

$$h(x_3) = E\, e^{-i k_{mn} x_3} + F\, e^{i k_{mn} x_3}$$

where E and F are constants and the axial wavenumber k_{mn} is

$$k_{mn} = \sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{n}{R}\right)^2 - \left(\frac{m\pi}{b-a}\right)^2}$$ [60%]

and n is the circumferential mode number.

d) Modes are 'cut-off' if k_{mn} is purely imaginary since they then decay exponentially along the duct, i.e. if

$$\frac{\omega^2}{c_0^2} < \left(\frac{n}{R}\right)^2 + \left(\frac{m\pi}{b-a}\right)^2$$ [10%]

For $\omega < nc_0/R$, the $m = 0$ and all higher m modes are cut-off.

e) A rotor with angular velocity Ω produces disturbances which are functions of $\Omega t - \theta$. Hence $\omega/n = \Omega$.

The condition for 'cut-off' disturbances requires

$$\frac{\omega}{n} = \Omega < c_0/R$$

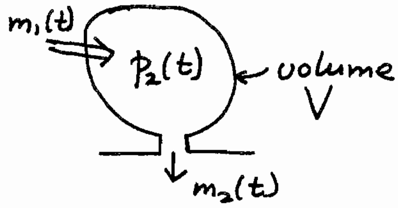
i.e. $\Omega R < c_0$

That is the fan tip speed must be <u>subsonic</u> if rotor alone noise is to be cut-off. [10%]

Module 4A14 2008

Solutions

Qu 2. a) Conservation of mass in the plenum:



$$V \frac{d\rho_2}{dt} = m_1 - m_2$$

For linear isentropic disturbances,

$$p_2' = c_0^2 \rho_2' .$$

Hence $\dfrac{V}{c_0^2} \dfrac{dp_2'}{dt} = m_1 - m_2 .$

The duct flow is given to be such that $p_2' = \dfrac{1}{K} \dfrac{dm_2}{dt}$

and hence we obtain $\dfrac{V}{c_0^2 K} \dfrac{d^2 m_2}{dt^2} = m_1 - m_2 .$

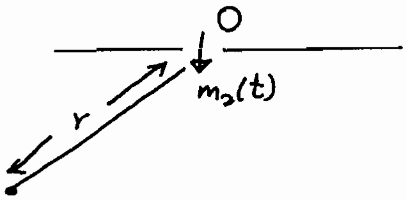
For disturbances of radian frequency ω,

$$-\frac{V}{c_0^2 K} \omega^2 m_2' = m_1' - m_2'$$

i.e. $$m_2' \left(1 - \frac{V\omega^2}{c_0^2 K} \right) = m_1'$$

or $$m_2'(t) = \frac{m_1'(t)}{1 - V\omega^2/(Kc_0^2)}$$ [30%]

b)



Mass ejection at a rate $m_2(t)$ is a monopole point source and in free space would generate a sound field

$p'(r,t) = \dfrac{\dot{m}_2(t - r/c)}{4\pi r}$ at a distance r from the source.

Here, the source is on large rigid surface and so there is an equal image source, resulting in a doubling of sound pressure :

$$p'(r,t) = \frac{2 \times \dot{m}_2(t - r/c)}{4\pi r}$$ [20%]

Module 4A14 2008

Solutions

Qu 2 cont.)

c) To determine the far-field sound we just need to combine results from a) and b). For fluctuations of frequency ω

$$p_{rms}(r) = \frac{\omega}{2\pi r} m_{2\,rms} = \frac{\omega \; m_{1\,rms}}{2\pi r\left(1 - V\omega^2/(\ell c_0^2)\right)}$$

Substituting the given values gives

$$p_{rms}(200) = \frac{2\pi\, 500}{2\pi\, 200}\; 2\times10^{-4}\; \frac{1}{1 - \frac{4.8\times10^{-4}}{0.04}\left(\frac{2\pi\,500}{343}\right)^2}$$

$$= 5\times10^{-4}\;\frac{1}{1-1.006683} = 0.0748\ \text{N/m}^2$$

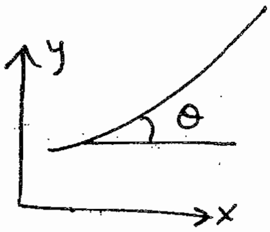
$= \underline{71.5\ \text{dB}}$ [40%]

d) The problem with the volume of $4.8\times10^{-4}\ \text{m}^3$ is that the plenum and short exit duct with $\ell = 0.04\text{m}$ are resonant at 500Hz. We want to move the volume away from this resonance while still keeping it small enough for the compact approximation of a uniform pressure fluctuation throughout the plenum (otherwise there might be additional resonances from standing waves in the plenum). For example, for $V = 1.5\times$ original volume,

$$p_{rms} = 5\times10^{-4}\;\frac{1}{1-1.5\times1.006683} = 9.8\times10^{-4}\ \text{N/m}^2$$

$= \underline{33.8\ \text{dB}}$

3(a) (i)



Snell: $\dfrac{\sin\theta}{c(x)} = \text{constant}$

suppose launched from $(0,0)$ at angle θ_0

$$y' = \tan\theta \,, \quad \frac{\dfrac{y'}{\sqrt{1+y'^2}}}{\alpha x+\beta} = \frac{\sin\theta_0}{\beta}$$

$$(y')^2\left\{1 - \frac{\sin^2\theta_0}{\beta^2}(\alpha x+\beta)^2\right\} = \frac{\sin^2\theta_0}{\beta^2}(\alpha x+\beta)^2$$

$$\rightarrow \quad y' = \frac{\left(\dfrac{\alpha x+\beta}{\beta}\right)\sin\theta_0}{\left[1 - \dfrac{(\alpha x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2}}$$

$$\int dy = \int \frac{\dfrac{\alpha x+\beta}{\beta}\sin\theta_0}{\left[1-\dfrac{(\alpha x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2}}\,dx$$

$$y = -\frac{\beta}{\alpha\sin\theta_0}\left[1 - \frac{(\alpha x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2} + k$$

when $x=0$ $y=0$ $\therefore$ $k = \dfrac{\beta}{\alpha\sin\theta_0}\left[1 - \dfrac{\beta^2\sin^2\theta_0}{\beta^2}\right] = \dfrac{\beta}{\alpha\tan\theta_0}$

$$\therefore y - \frac{\beta\cot\theta_0}{\alpha} = \frac{-\beta}{\alpha\sin\theta_0}\left[1 - \frac{(\alpha x+\beta)^2}{\beta^2}\sin^2\theta_0\right]^{1/2}$$

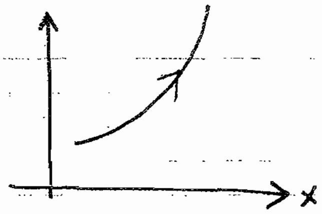
$$\rightarrow \left(x+\frac{\beta}{\alpha}\right)^2 + \left(y - \frac{\beta\cot\theta_0}{\alpha}\right)^2 = \frac{\beta^2}{\alpha^2\sin^2\theta_0}$$

$\therefore$ circle centre $-\dfrac{\beta}{\alpha}, \dfrac{\beta\cot\theta_0}{\alpha}$

radius $\left|\beta/\alpha\sin\theta_0\right|$. 6

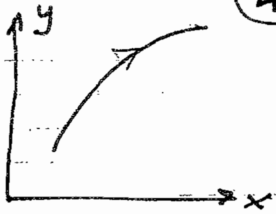
Qu 3(a)(i) cont.

$\alpha > 0$



rays bend up

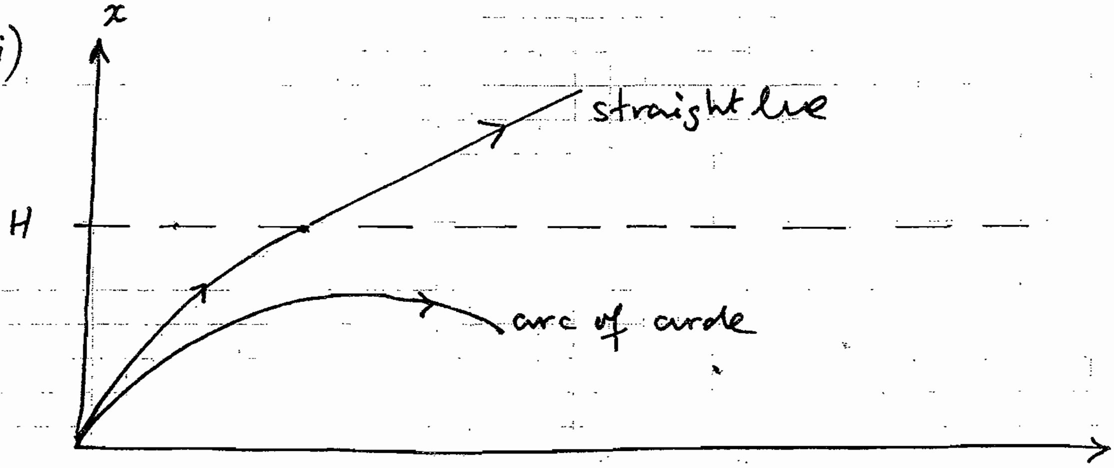
$\alpha < 0$



rays bend down.

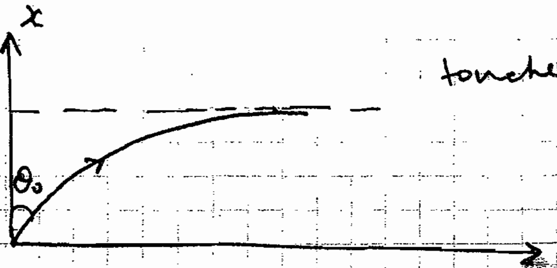
1

(ii)



Here must have α positive to make sense

critical case:



touches line $y = H$

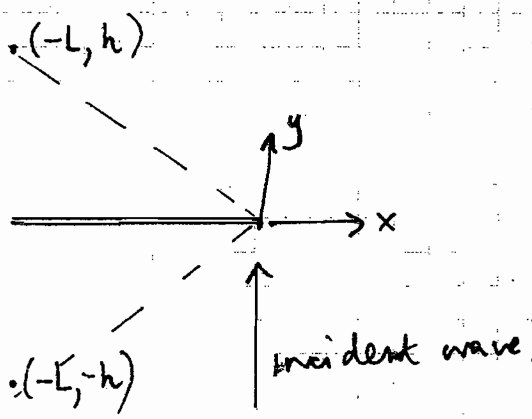
$$\frac{\sin\theta}{c_0} = \text{constant}$$

$$\therefore \frac{\sin\theta_0}{\alpha H} = \frac{\sin(\pi/2)}{2\alpha H}$$

$$\therefore \sin\theta_0 = 1/2, \quad \theta_0 = \sin^{-1}(1/2) = 30^\circ.$$

4

3(b)



Qu 3b cont.)

In notation of data sheet $\theta_0 = -\pi/2$

(i) $\theta = \pi - \tan^{-1}(h/L)$

$$p_d = \underset{=1}{\underset{\uparrow}{p_{inc}}} \left(\frac{2}{\pi k_0 r}\right)^{1/2} \frac{\sin\left(-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{h}{L}\right)\right)}{\cos\left(\pi-\tan^{-1}\left(\frac{h}{L}\right)\right)+\cos(-\pi/2)} e^{-\frac{i\pi}{4}-ik_0 r}$$

$r = \sqrt{h^2+L^2}$

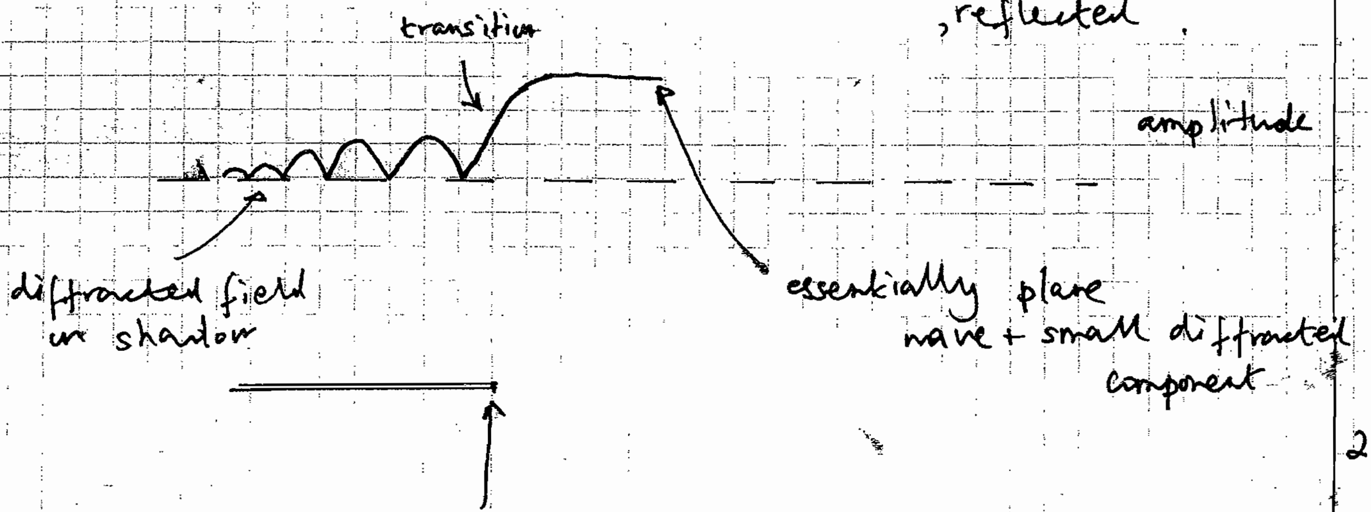
$$\rightarrow \underline{\underline{\left(\frac{2}{\pi k_0 r}\right)^{1/2} \frac{+1}{\sqrt{2}} \frac{\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{h}{L}\right)\right)}{\cos\left(\tan^{-1}\left(\frac{h}{L}\right)\right)} e^{-i\pi/4-ik_0 r}}}.$$

(ii) $\theta = -\pi + \tan^{-1}(h/L)$

$$\therefore p_d = \underline{\underline{-\left(\frac{2}{\pi k_0 r}\right)^{1/2} \frac{1}{\sqrt{2}} \frac{\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{h}{L}\right)\right)}{\cos\left(\tan^{-1}(h/L)\right)} e^{-\frac{i\pi}{4}-ik_0 r}}}$$

In case (i) geometrical optics field $=0$ (shadow)

In (ii) " " " $= \exp(i\omega t + ik_0 y)$, reflected.



2

4 (a) Starting from $p'(\underline{x},t) = \dfrac{\partial^2}{\partial x_i \partial x_j} \displaystyle\int \dfrac{T_{ij}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right)}{|\underline{x}-\underline{y}|} d^3y$

(data sheet)

Replace $\dfrac{1}{|\underline{x}-\underline{y}|}$ by $\dfrac{1}{|\underline{x}|}$, as assume observer in far field

further, assume retarded-time variation negligible across source – compact

$$\therefore p'(\underline{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{T_{ij}\left(\underline{y}, t - \frac{|\underline{x}|}{c_0}\right)}{|\underline{x}|} d^3y .$$

Take $\dfrac{1}{|\underline{x}|}$ outside derivatives, as only want $\dfrac{1}{|\underline{x}|}$ term

in far field

$$p' \sim \frac{1}{|\underline{x}|} \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{T_{ij}\left(\underline{y}, t - \frac{|\underline{x}|}{c_0}\right)}{|\underline{x}|} d^3y$$

$$= \frac{1}{|\underline{x}|} \frac{\partial^2}{\partial x_i \partial x_j} S_{ij}\left(t - |\underline{x}|/c_0\right), \quad S_{ij} \equiv \int T_{ij} .$$

$$= \frac{1}{|\underline{x}|} \frac{x_i x_j}{|\underline{x}|^2} \frac{1}{c_0^2} \ddot{S}_{ij}$$

Take $T_{ij} = O(\rho u_0^2)$ $\quad u_0$ – velocity scale

$S_{ij} = \int T_{ij} = O(\rho_0 u_0^2 \ell^3)$ $\quad \ell$ = length scale

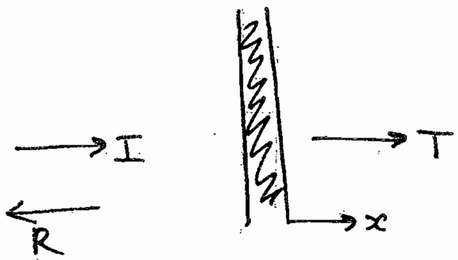
time derivative $= O(u_0/\ell)$, ie inverse time scale

$$\therefore p' = O\left(\frac{1}{|\underline{x}|} \cdot \frac{1}{c_0^2} \frac{u_0^2}{\ell^2} \rho_0 u_0^2 \ell^3\right)$$

$$= O\left(\frac{\ell}{|\underline{x}|} \rho_0 c_0^2 \left[\frac{u_0}{c_0}\right]^4\right)$$

Acoustic intensity $\propto p'^2 \propto \left[\dfrac{u_0}{c_0}\right]^8$.

Qu 4(b)



$$x<0 \quad p' = I e^{i\omega t - i\omega x/c_0} + R e^{i\omega t + i\omega x/c_0}$$

$$x>0 \quad p' = T e^{i\omega t - i\omega x/c_0}$$

Pressure difference across the wall drives wall motion.

Eq^n. of motion is

$$m \frac{d}{dt}\left(V e^{i\omega t}\right) = p'(x=-0,t) - p'(x=+0,t)$$

$$\Rightarrow \quad m i\omega V = -T + I + R. \qquad (1)$$

Acoustic velocity at $x=+0$ is $\dfrac{T}{\rho_0 c_0}$ ← plane wave impedance

" " " $x=-0$ is $\dfrac{I-R}{\rho_0 c_0}$

Continuity $\Rightarrow \quad \dfrac{T}{\rho_0 c_0} = \dfrac{I-R}{\rho_0 c_0} = V \qquad (2)$

put (2) into (1) $\quad m i\omega V = \rho_0 c_0\left[-V\right] + I \quad + I - \rho_0 c_0 V$

$$\Rightarrow \quad V = \frac{2I}{m i\omega + 2\rho_0 c_0}$$

$$(2) \Rightarrow \quad T = \underline{\frac{2\rho_0 c_0 I}{2\rho_0 c_0 + m i\omega}}$$

$$\therefore \quad \frac{|T|}{|I|} = \underline{\frac{2\rho_0 c_0}{\sqrt{m^2\omega^2 + 4\rho_0^2 c_0^2}}}$$

Qu 4b) cont.

$$\frac{|T|}{|I|} = \frac{2}{\left[\left(\frac{m\omega}{\rho_0 c_0}\right)^2 + 4\right]^{1/2}}$$

one wavelength is $\frac{2\pi c_0}{\omega}$, mass of air is $\frac{2\pi c_0 \rho_0}{\omega}$ per unit wall area

$$\therefore \frac{\text{mass of wall per unit area}}{\text{mass of sound in one } \lambda \text{ " " "}} = \left(\frac{m\omega}{\rho_0 c_0}\right) \frac{1}{2\pi}$$

Hence $\frac{m\omega}{\rho_0 c_0}$ controls level of transmission

$\frac{m\omega}{\rho_0 c_0} \gg 1 \Rightarrow$ heavy wall, long waves $|T| \ll |I|$

$\frac{m\omega}{\rho_0 c_0} \ll 1 \Rightarrow$ light wall, short waves $|T| \approx |I|$. 2

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