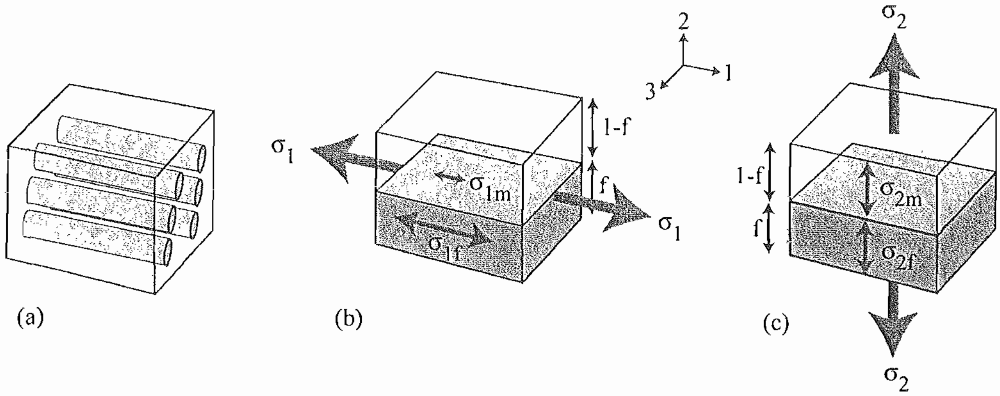
Q1/1

Module 4C2, Designing with Composites – Crib 2007/2008

1. (a) To predict the axial and transverse stiffness of an aligned composite, both constituents, the matrix and the fibres, are represented as parallel slabs bonded together, with thicknesses in proportion to their volume fractions.



When the stress is applied in the 1-direction (see Figure b), an equal strain condition applies

$$\varepsilon_1 = \varepsilon_{1f} = \frac{\sigma_{1f}}{E_f} = \varepsilon_{1m} = \frac{\sigma_{1m}}{E_m}$$

where the subscripts *f* and *m* refer to the reinforcement and matrix respectively.

The applied stresses σ_1 will be shared by the matrix and the fibres according to their volume fraction in the composite

$$\sigma_1 = f\,\sigma_{1f} + (1\text{-}f)\,\sigma_{1m}$$

where *f* is the fibre volume fraction.

The Young's modulus of the composite can be written

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{f\sigma_{1f} + (1-f)\sigma_{1m}}{\dfrac{\sigma_{1f}}{E_f}} = fE_f + (1-f)E_m$$

When the stress is applied in the 2 direction (see Figure c), the stiffness can be predicted using an equal stress assumption

$$\sigma_2 = E_2\,\varepsilon_2 = \sigma_{2f} = E_f\,\varepsilon_{2f} = \sigma_{2m} = E_m\ \varepsilon_{2m}$$

The overall net strain can be written as

$$\varepsilon_2 = f\,\varepsilon_{2f} + (1\text{-}f)\,\varepsilon_{2m}$$

The transverse Young's modulus of the composite can now be written

$$E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{\sigma_{2f}}{f\varepsilon_{2f} + (1-f)\varepsilon_{2m}} = \left[\frac{f}{E_f} + \frac{1-f}{E_m}\right]^{-1}$$

For 60 vol% glass fibres in a polyester matrix

$$E_1 = 0.6 \times 76 + 0.4 \times 3.4 = 46.96 \text{ GPa}$$

$$E_2 = \left[\frac{0.6}{76} + \frac{0.4}{3.4}\right]^{-1} = 7.96 \text{ GPa}$$

1.(b)

$$\overline{S}_{16} = (2S_{11} - 2S_{12} - S_{66})c^3 s - (2S_{22} - 2S_{12} - S_{66})cs^3$$

where $c = \cos\theta$, $s = \sin\theta$

For $\overline{S_{16}}$ to be zero (no tensile-shear interactions):

$$(2S_{11} - 2S_{12} - S_{66})c^3 s = (2S_{22} - 2S_{12} - S_{66})cs^3$$

Two solutions are $\theta = 0°$ ($s = 0$) and $\theta = 90°$ ($c = 0$). Third solution is:

$$\frac{s^2}{c^2} = \frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}} \Rightarrow \theta = \tan^{-1}\left(\left(\frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}}\right)^{1/2}\right)$$

$$S_{11} = \frac{1}{E_1} = \frac{1}{47}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.24}{47}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{8}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{3}$$

$$\theta = \tan^{-1}\left(\left(\frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}}\right)^{1/2}\right) = \tan^{-1}(1.96) = 63°$$

(c)

(i) Calculate ν_{21}

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \Rightarrow \nu_{21} = 0.04$$

Calculate $[Q]$ in principal material axes (1, 2).

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{46}{1-0.24\times 0.04} = 47.46 \text{ GPa}$$
$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = \frac{8}{1-0.24\times 0.04} = 8.08 \text{ GPa}$$
$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{0.24\times 8}{1-0.24\times 0.04} = 1.94 \text{ GPa}$$
$$Q_{66} = G_{12} = 3 \text{ GPa} \qquad Q_{16} = Q_{26} = 0$$

$$[Q] = \begin{bmatrix} 47.46 & 1.94 & 0 \\ 1.94 & 8.08 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{GPa}$$

Calculate the transformed stiffness matrix $[\overline{Q}]$ in the global x-y axes. The transformed stiffness matrix for the 0° ply (1st ply) is given by

$$[\overline{Q}]_0 = Q$$

The transformed stiffness matrix for the 60° ply (2nd ply) is given by

$$\left(\overline{Q}_{11}\right)_{60°} = 47.46\, c^4 + 8.08\, s^4 + 2\left(1.94 + 2\times 3\right)s^2c^2 = 10.49 \text{ GPa}$$
$$\left(\overline{Q}_{12}\right)_{60°} = \left(47.46 + 8.08 - 4\times 3\right)s^2c^2 + 1.94\left(c^4 + s^4\right) = 9.37 \text{ GPa}$$
$$\left(\overline{Q}_{22}\right)_{60°} = 47.46\, s^4 + 8.08\, c^4 + 2\left(1.94 + 2\times 3\right)s^2c^2 = 30.18 \text{ GPa}$$
$$\left(\overline{Q}_{16}\right)_{60°} = \left(47.46 - 1.94 - 2\times 3\right)c^3s - \left(8.08 - 1.94 - 2\times 3\right)cs^3 = 4.23 \text{ GPa}$$
$$\left(\overline{Q}_{26}\right)_{60°} = \left(47.46 - 1.94 - 2\times 3\right)cs^3 - \left(8.08 - 1.94 - 2\times 3\right)c^3s = 12.82 \text{ GPa}$$
$$\left(\overline{Q}_{66}\right)_{60°} = \left(47.46 + 8.08 - 2\times 1.94 - 2\times 3\right)s^2c^2 + 3\left(s^4 + c^4\right) = 10.44 \text{ GPa}$$

where $c = \cos 60$, $s = \sin 60$

The transformed stiffness matrix for the 120° ply (3st ply) is

$$\left(\overline{Q}_{11}\right)_{120°} = \left(\overline{Q}_{11}\right)_{60°} = 10.49 \text{ GPa}$$
$$\left(\overline{Q}_{12}\right)_{120°} = \left(\overline{Q}_{12}\right)_{60°} = 9.37 \text{ GPa}$$
$$\left(\overline{Q}_{22}\right)_{120°} = \left(\overline{Q}_{22}\right)_{60°} = 30.18 \text{ GPa}$$
$$\left(\overline{Q}_{16}\right)_{120°} = -\left(\overline{Q}_{16}\right)_{60°} = -4.23 \text{ GPa}$$
$$\left(\overline{Q}_{26}\right)_{120°} = -\left(\overline{Q}_{26}\right)_{60°} = -12.82 \text{ GPa}$$
$$\left(\overline{Q}_{66}\right)_{120°} = \left(\overline{Q}_{66}\right)_{60°} = 10.44 \text{ GPa}$$

where $c = \cos 120,\ s = \sin 120$

Calculate the laminate extensional stiffness matrix $[A]$

$$A_{ij} = \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k (z_k - z_{k-1})$$

$$\begin{aligned} A_{11} &= 0.2 \cdot \left(\overline{Q}_{11}\right)_0 + 0.2 \cdot \left(\overline{Q}_{11}\right)_{60} + 0.2 \cdot \left(\overline{Q}_{11}\right)_{120} = \\ &= 0.2 \times 47.46 + 0.2 \times 10.49 + 0.2 \times 10.49 \\ &= 0.2 \text{ mm} \times (47.46 + 10.49 + 10.49) \text{ GPa} \\ &= 13.69 \text{ MNm}^{-1} \end{aligned}$$

Similarly

$$\begin{aligned} A_{12} &= 0.2 \text{ mm} \times (1.94 + 9.37 + 9.37) \text{ GPa} \\ &= 4.14 \text{ MNm}^{-1} \\ A_{22} &= 0.2 \text{ mm} \times (8.08 + 30.18 + 30.18) \text{ GPa} \\ &= 13.69 \text{ MNm}^{-1} \\ A_{16} &= 0.2 \text{ mm} \times (0 + 4.23 - 4.23) \text{ GPa} \\ &= 0 \\ A_{26} &= 0.2 \text{ mm} \times (0 + 12.82 - 12.82) \text{ GPa} \\ &= 0 \\ A_{66} &= 0.2 \text{ mm} \times (3 + 10.44 + 10.44) \text{ GPa} \\ &= 4.77 \text{ MNm}^{-1} \end{aligned}$$

$$[A] = \begin{bmatrix} 13.69 & 4.14 & 0 \\ 4.14 & 13.69 & 0 \\ 0 & 0 & 4.77 \end{bmatrix} \text{MNm}^{-1}$$

Q 1 (contd.)

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The $[A]$ matrix is isotropic in form:

$$[A]=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & (A_{11}-A_{12})/2 \end{bmatrix}$$

(ii) The laminate is balanced ($A_{16} = A_{26} = 0$). A balanced laminate is one in which the laminate as a whole exhibits no tensile-shear interactions, for any loading angle. Tensile-shear interactions (tensile (normal) strains resulting from applied shear stresses and vice-versa) lead to distortions. These can be minimised if the stacking sequence is chosen carefully, since the distortions from individual laminae may then cancel each other out.

Q 2 (a) Prepreg
- old or incorrectly stored
- damaged during cutting
- unclean environment leading to contamination

Layup
- poor alignment or ply placement
- inclusion of bubbles between plies
- generation of fibre waviness

Curing
- incorrect cure cycle
- development of residual stresses during curing

(b)
Low cost yacht.

Materials
Need reasonable mechanical performance, probably strength and stiffness implies composites are desirable. Low cost rules out CFRP in favour of GFRP.
Impact toughness - could use a woven material.

Manufacture
Spray layup would give a low cost, but at the cost of reduced performance. Resin infusion techniques would give a better performance but at higher cost. Hand lay-up might be appropriate for one-offs, but the labour costs are likely to be high for a reasonable production run. Low cost will be helped by making much of the hull in one piece. Probably spray layup would be a preferred route, with perhaps hand layup to strengthen critical areas.

High performance yacht.
Materials
Need to minimise weight, with cost being less critical implies use of CFRP, probably in unidirectional prepreg form. For a lower performance racing yacht GFRP might be appropriate. Sandwich structures will help reduce weight.

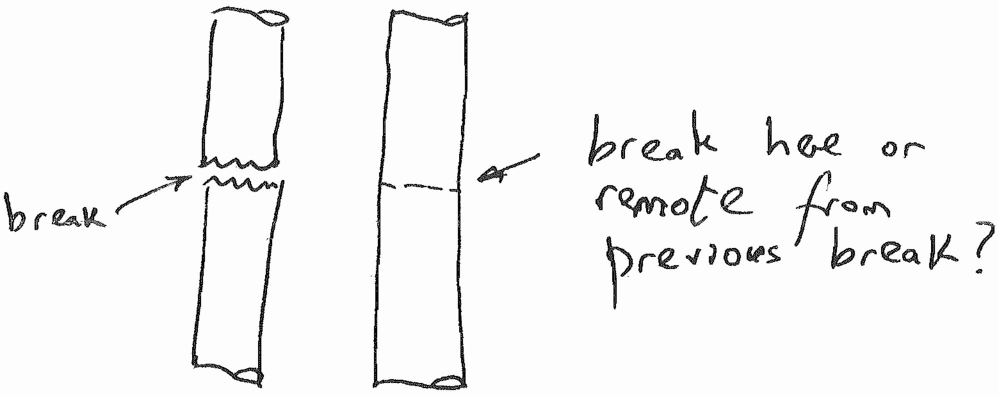
Manufacture
Ideally would autoclave, but may be difficult to have a large enough autoclave (depending on budget). Alternatively use vacuum resin injection to give reasonable properties. In the later case it might be worth making some smaller critical parts from an autoclaved route and bonding them.

See notes for descriptions of process routes.

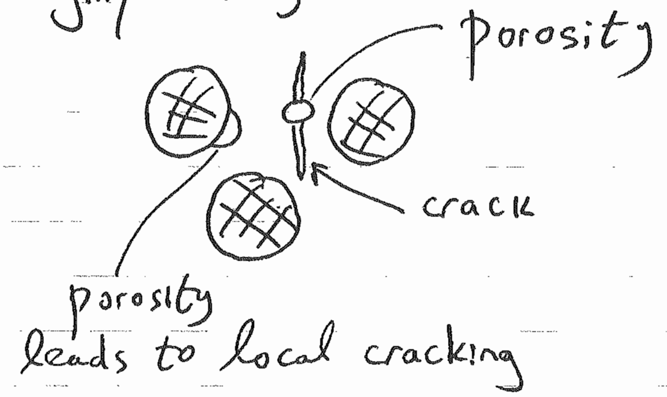
(c) Testing will be critical at all stages. Start with testing of coupons to establish properties of the material, including some form of damage tolerance (e.g. impact tests). Then test sub-structures, e.g. features, load attachment points, stiffeners. May also need to check that the manufacturing route is giving appropriate properties by manufacturing demonstrator parts. Finally test some subsections of the yacht (probably not feasible to test the whole hull). Need to check during the manufacturing stage that the structure is being built to give the right properties as per the design. Non destructive testing will help, or testing of appropriate parts made during the manufacturing cycle.

Q3. (a) (i) The longitudinal strength is primarily controlled by fibre properties.

Adjacent porosity can produce a stress concentration, resembling an elliptical hole, and give a slight reduction in strength.



(ii) The transverse tensile strength is dominated by failure in matrix. Expect porosity to give local stress concentrations in the matrix and to reduce strength significantly.



[25%]

Q3(b)(i) First, find laminate strain.

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix} = A^{-1}\begin{pmatrix}N_x\\ N_y\\ N_{xy}\end{pmatrix} = A^{-1}\begin{pmatrix}11\sigma t\\ 0\\ 0\end{pmatrix}$$

$$= 11\times10^{-12}\begin{pmatrix}2.36\\ -0.62\\ 0\end{pmatrix}\sigma \quad N^{-1}m^2$$

Now, find the ply strain in material co-ordinates:

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix} = T^{-T}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$$

where $\cos\theta \approx 1-\frac{1}{2}\theta^2$
$\sin\theta \approx \theta$

$$\Rightarrow \quad T^{-T} \approx \begin{pmatrix}1-\theta^2 & \theta^2 & \theta\\ \theta^2 & 1-\theta^2 & -\theta\\ -2\theta & 2\theta & 1-2\theta^2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{pmatrix} = \begin{pmatrix}2.36 - 2.98\theta^2\\ -0.62+2.98\theta^2\\ -5.96\theta\end{pmatrix} 11\times10^{-12}\sigma \quad N^{-1}m^2$$

Maximum strain failure criterion is

$$\varepsilon_{11} \leq \frac{S_L^+}{E_1} = \frac{1103}{39\times10^3} = 0.0283$$

$$\Rightarrow \quad \sigma_c = \frac{0.0283}{(2.36-2.98\theta^2)\ 11\times10^{-12}}$$

Q 3(b)(i) cont

$$\Rightarrow \sigma_c = (1089 + 1367\theta^2) \text{ MPa}$$ [30%]

(ii) To use the Tsai-Hill failure criterion we need to find the stresses in the 0 ply

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = [Q] \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} 39 & 2.2 & 0 \\ 2.2 & 8.4 & 0 \\ 0 & 0 & 4.1 \end{pmatrix} \begin{pmatrix} 2.36 \\ -0.62 \\ -5.96\theta \end{pmatrix} 11 \times 10^{-3} \sigma$$

obtained by putting $\theta = 0$ in $\bar{Q}_\theta$ expression

$$\Rightarrow \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} 90.7 \\ -0.02 \\ -24.4\theta \end{pmatrix} \times 10 \times 10^{-3} \sigma$$

Tsai - Hill : $S_L^+ = 1103$ MPa, $S_T^- = 138$ MPa, $S_{LT} = 82.7$ MPa

$$\left(\frac{\sigma_1}{S_L}\right)^2 + \left(\frac{\sigma_2}{S_T}\right)^2 - \frac{\sigma_1 \sigma_2}{S_L^2} + \frac{\tau_{12}}{S_{LT}^2} \geq 1$$

$$\Rightarrow \left(\frac{90.7}{1103}\right)^2 + \left(\frac{24.4\theta}{82.7}\right)^2 \geq \frac{1}{(11 \times 10^{-3} \sigma)^2} \quad \frac{1}{(\text{MPa})^2}$$ (neglecting terms with σ_2)

$$\sigma_{\text{fc}}^2 \leq \frac{1103^2}{1 + 12.8\theta^2}$$

$$\Rightarrow \sigma < 1103(1 - 12.8\theta^2) \text{ MPa}$$

[35%]

Neither criterion is very accurate. The maximum strain criterion is consistent with the physical notion of tensile fibre failure. But it is likely that the presence of shear will reduce the fibre strength.

The Tsai-Hill curve tends to fit the data better at large θ, but may not be so accurate at small θ.

Note that the two criteria predict opposite sensitivity to θ.

[10%]

4. (a) $q = N_{xy} = \frac{'T'}{2Ae} = \frac{Q}{2b^2} = N_{x'y'}$

Torsion does not generate stress in N_x or N_y directions.

Now apply a moment M.

$\frac{\sigma}{y} = \frac{M}{I}$ with $I = \frac{1}{12}b^4 - \frac{1}{12}(b-2t)^4$

$\Rightarrow I = \frac{2}{3}b^3 t$.

At A, $y=0$ and $N_{x'} = N_{y'} = 0$ as before.

At B, $y = b/2$, $N_x = \sigma . t = \frac{M t\, b/2}{\frac{2}{3}b^3 t}$

$\Rightarrow N_x = \frac{3}{4}\frac{M}{b^2}$, $N_y = 0$. [20%]

(b) (i) $N_{x'y'} = \frac{Q}{2b^2} = \frac{90 \times 10^3}{2 \times 0.4^2}$ N/m = 281 kN/m

Use strain allowables, from Table 1 of the Module 4C2 Data sheet.

$e_{LT} = \gamma = 0.005 = \frac{N_{x'y'}}{Gt}$

$\Rightarrow Gt = 56 \text{ MNm}^{-1}$.

First iteration, assume mainly ±45° plies, put 10% 0° plies and 10% 90° plies.

$\Rightarrow G = 30$ GPa $\Rightarrow t = 1.87$ mm
= 15 plies.

4(b)(i) contd.

3 plies are either 0° or 90°,
so choose 2 plies at 0°
1 ply at 90°
12 plies at 45°

eg. $[(\pm 45)_3/0/\overline{90}]_s$ [30%]

4.(b)(ii) Again, $Gt_{xy} = 56 \text{ MNm}^{-1}$

Also, $e^+ = 0.004 = \frac{N_x}{Et}$ where $N_x = \frac{3}{4}\frac{M}{b^2}$

Hence, $Et = \frac{3}{4}\frac{M}{b^2 e^+} = \frac{3}{4}\frac{300}{0.04^2 \times 0.004}$

$\Rightarrow Et = 35.2 \text{ MNm}^{-1}$

With just 0° fibres : $E = 140$ GPa
$\Rightarrow t = 0.251$ mm $= 2$ plies.

So, the same lay-up as above is adequate : $[(\pm 45)_3/0/\overline{90}]_s$.

[50%]