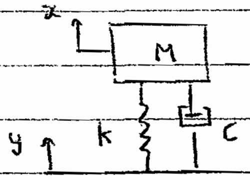
2008 IIB

MODULE 4C7 : RANDOM AND NON-LINEAR VIBRATIONS . EXAM CRIBS 2008 PROF RS LANGLEY

1) a)



$M\ddot{x} + C(\dot{x}-\dot{y}) + k(x-y) = 0$

or $\ddot{x} + 2\beta\omega_n(\dot{x}-\dot{y}) + \omega_n^2(x-y) = 0$ $\left\{ \begin{array}{l} \omega_n^2 = k/M \\ 2\beta\omega_n = C/M \end{array} \right.$

Put $r = x-y$ $\Rightarrow$ $\underline{\ddot{r} + 2\beta\omega_n\dot{r} + \omega_n^2 r = -\ddot{y}}$

For ω_n within spectral bandwidth, $S_{\ddot{y}\ddot{y}}(\omega_n) = S_0$ ← white noise approximation.

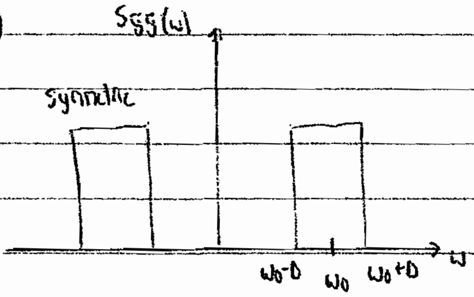
So, standard results for white noise excitation : $\underline{\sigma_r^2 = \dfrac{\pi S_0}{2\beta\omega_n^3}}$ $\underline{\sigma_{\dot{r}}^2 = \dfrac{\pi S_0}{2\beta\omega_n}}$

Power dissipated = Force x velocity = $C\dot{r} \times \dot{r}$ $\Rightarrow$ $E[P] = C\sigma_{\dot{r}}^2 = \dfrac{C\pi S_0}{2\beta\omega_n} = \underline{M\pi S_0}$ [30%]

b) Low damping ⇒ high response but low C $\Big\}$ Mean square response $\propto \frac{1}{C}$ implies power
High damping ⇒ low response but high C $\Big\}$ is independent of C

For C=0 the response cannot be stationary and hence result does not hold. For $C\to 0$ the result will break down due to onset of non-linearity introduced by large response [20%]

c)



$S_{\dot{y}\dot{y}}(\omega) = \dfrac{1}{\omega^2} S_{\ddot{y}\ddot{y}}(\omega)$

$\sigma_{\dot{y}}^2 = \displaystyle\int_{-\infty}^{\infty} S_{\dot{y}\dot{y}}(\omega)d\omega = 2\int_{\omega_0-\Delta}^{\omega_0+\Delta} \frac{1}{\omega^2} S_0 d\omega$

$\sigma_{\dot{y}}^2 = 2\left[\dfrac{1}{\omega_0-\Delta} - \dfrac{1}{\omega_0+\Delta}\right] = \dfrac{4\Delta}{\omega_0^2-\Delta^2}$

Power dissipated by C_d = $C_d\sigma_{\dot{y}}^2 = C_d\dfrac{4\Delta}{\omega_0^2-\Delta^2}$ → place equal to $M\pi S_0$

$\Rightarrow$ $\underline{C_d = \dfrac{M\pi S_0}{4\Delta}(\omega_0^2-\Delta^2)}$ [25%]

d) Probability that ω_n falls within $\omega_0 - \Delta$ to $\omega_0 + \Delta$ is $\frac{2\Delta}{\omega_0 + \Delta}$ ← interval / range of ω_n's

Number of oscillators with ω_n in spectrum $= N \times \frac{2\Delta}{\omega_0 + \Delta}$

If ω_n is outside spectrum then response is small and power dissipation is negligable

Thus $E[P] = M\pi S_0 \times \frac{2\Delta N}{\omega_0 + \Delta}$

↑ From one absorber ↑ number of absorbers

$$\Rightarrow \quad C_a = \frac{M\pi S_0}{4\Delta}(\omega_0^2 - \Delta^2) \times \left(\frac{2\Delta N}{\omega_0 + \Delta}\right) = \frac{M\pi S_0}{2} \underbrace{\left(\frac{N}{\omega_0 + \Delta}\right)}_{\text{"Modal density"}} (\omega_0^2 - \Delta) \qquad [25\%]$$

2) a) Gap $r = x - y$ — For standard deviations assume all variables are adjusted to zero mean, i.e. $r =$ gap - mean value

$$r^2 = x^2 + y^2 - 2xy \Rightarrow \sigma_r^2 = \sigma_x^2 + \sigma_y^2 - 2E[xy] = \underline{\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y}$$

Similarly $\underline{\sigma_{\dot r}^2 = \sigma_{\dot x}^2 + \sigma_{\dot y}^2 - 2\rho_{\dot x \dot y}\sigma_{\dot x}\sigma_{\dot y}}$ [25%]

b) $\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot r}}{\sigma_r}\right) e^{-\frac{1}{2}(b/\sigma_r)^2}$ with σ_r and $\sigma_{\dot r}$ as above.

For $\rho_{xy} \to 1$, $\sigma_r \to 0$. The term $\frac{1}{\sigma_r} \to \infty$, but the exponential $\to 0$ at a much faster rate

$\Rightarrow$ $\underline{\nu_b^+ \to 0 \text{ as } \rho_{xy} \to 1}$

This makes physical sense: x and y are fully correlated and $x-y$ never changes [25%]

c) Need to maximise σ_r $\Rightarrow$ $\underline{\rho_{xy} = -1}$ (bounded between -1 and 1, hence this is the lowest value). [10%]

d) $\sigma_r^2 = 2^2 + 4^2 - 2 \times 0.7 \times 2 \times 4 = 8.8 \Rightarrow \sigma_r = 2.96$

$\sigma_{\dot r}^2 = 0.6^2 + 1^2 - 2 \times 0.6 \times 1 \times 0.4 = 0.88 \Rightarrow \sigma_{\dot r} = 0.9381$

$\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot r}}{\sigma_r}\right) e^{-\frac{1}{2}(b/\sigma_r)^2}$ with $b = 12$ $\Rightarrow$ $\nu_b^+ = 1.504 \times 10^{-5}$

Failure probability $P_f = 1 - e^{-\nu_b^+ T} = 1 - e^{-\nu_b^+ \times 3 \times 60 \times 60} = \underline{0.1507}$ [20%]

e) Factors: storm statistics
consequences of sea hitting deck - danger, cost of damage
cost of reducing probability
confidence in analysis
etc

Q3.
$$\ddot{x} - \varepsilon(1-x^4)\dot{x} + x = 0 \quad ; \quad 0 < \varepsilon \ll 1$$

zero order (set $\varepsilon = 0$)

$$\ddot{x}_0 + x_0 = 0$$

$$x_0 = A\cos t$$

1st order

$$\ddot{x}_1 + x_1 = \varepsilon(1-x_0^4)\dot{x}_0$$

$$\ddot{x}_1 + x_1 = \varepsilon(1 - A^4\cos^4 t)(-A\sin$$

$$\ddot{x}_1 + x_1 = -A\varepsilon\left(1 - A^4\left[\frac{3 + 4\cos 2t + \cos 4t}{8}\right]\right)\sin t$$

$$= -A\varepsilon \sin t + \frac{A^5\varepsilon}{8}(3\sin t)$$
$$+\frac{A^5\varepsilon}{8}\left(4\cos 2t \sin t + \cos 4t \sin t\right)$$

$$= -A\varepsilon\sin t + \frac{A^5\varepsilon}{8}\cdot 3\sin t$$
$$+\frac{A^5\varepsilon}{8}\left(\frac{4}{2}(\sin 3t - \sin t) + \frac{1}{2}(\sin 5t - \sin 3t)\right)$$

$$= \sin t\left(-A\varepsilon + \frac{1}{8}A^5\varepsilon\right) + \frac{A^5\varepsilon}{8}\sin 3t\left(\frac{3}{2}\right)$$
$$+\frac{A^5\varepsilon}{16}\sin 5t \qquad \text{—(1)}$$

For a P.I. try:

$$x_1 = \alpha\sin 3t + \beta\sin 5t + \gamma t\cos t$$

$$\dot{x}_1 = 3\alpha\cos 3t + 5\beta\cos 5t + \gamma\cos t - \gamma t\sin t$$
$$\ddot{x}_1 = -9\alpha\sin 3t - 25\beta\sin 5t - 2\gamma\sin t - \gamma t\cos t$$

$$\ddot{x}_1 + x_1 = -8\alpha\sin 3t - 24\beta\sin 5t - 2\gamma\sin t \quad (2)$$

equating (1) and (2) :

$$\alpha = \frac{-3A^5\varepsilon}{128}$$

$$\beta = \frac{-A^5\varepsilon}{384}$$

$$\gamma = \frac{A\varepsilon}{2} - \frac{A^5\varepsilon}{16}$$

solution to first order is found.

(ii) For a steady amplitude solution, the secular term in $\sin t$ must vanish or :-

$$\frac{A^5\varepsilon}{8} = A\varepsilon$$

$$A^4 = 8$$

$$A = \sqrt[4]{8} = 1.68.$$

Q4

(a) $\ddot{x} + \left(x - \frac{\alpha}{\beta - x}\right) = 0$

Transform to two first order differential equations

$$\dot{x} = y$$
$$\dot{y} = -x + \frac{\alpha}{\beta - x}$$

equilibrium points: $y = 0$ and $x = \frac{\alpha}{\beta - x}$

$$x(\beta - x) - \alpha = 0$$
$$x^2 - \beta x + \alpha = 0$$

Roots $\quad x_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha}}{2}$

Equilibrium points exist for

$$\alpha < \beta^2/4$$

(b) Linearise about equilibrium point

$$x = x_{\pm} + \varepsilon_1$$
$$y = \varepsilon_2$$
$$\dot{\varepsilon}_1 = \varepsilon_2$$
$$\dot{\varepsilon}_2 \simeq -\varepsilon_1 + \frac{\alpha\ \varepsilon_1}{(\beta - x_{\pm})^2}$$

The eigenvalues (λ) are given by (for characteristic matrix A)

$$\lambda^2 + 1 - \frac{\alpha}{(\beta - x_\pm)^2} = 0$$

$$\lambda = \pm \sqrt{\frac{\alpha}{(\beta - x_\pm)^2} - 1}$$

For ~~eigenvalue~~ equilibrium point $(0, x_+)$ it can be shown that $\frac{\alpha}{(\beta - x_+)^2} - 1 > 0$

$\therefore$ saddle point

equilibrium point $(0, x_-)$ is a centre as $\frac{\alpha}{(\beta - x_-)^2} - 1 < 0$.

(iii)