2008 Part IIB - 4C9 - CONTINUUM MECHANICS PROF JA WILLIAMS

1 (i) To prove $(\underline{a}\times\underline{b})\cdot(\underline{c}\times\underline{d}) = \underline{a}\cdot\underline{c}\ \underline{b}\cdot\underline{d} - \underline{a}\cdot\underline{d}\ \underline{b}\cdot\underline{c}$

$$\underline{a}\times\underline{b} = e_{ijk}a_j b_k = x_i$$

$$\underline{c}\times\underline{d} = e_{lmn}c_m d_n = y_l$$

$$\underline{x}\cdot\underline{y} = x_i y_i = e_{ijk}a_j b_k\, e_{imn}c_m d_n$$

$$\Rightarrow e_{ijk}e_{imn}a_j b_k c_m d_n$$

$$\Rightarrow (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_j b_k c_m d_n$$

$$\Rightarrow \delta_{jm}\delta_{kn}a_j b_k c_m d_n - \delta_{jn}\delta_{km}a_j b_k c_m d_n$$

$$\Rightarrow \delta_{kn}a_j b_k c_j d_n - \delta_{km}a_j b_k c_m d_j$$

$$\Rightarrow a_j b_k c_j d_k - a_j b_k c_k d_j$$

$$\Rightarrow a_j c_j b_k d_k - a_j d_j b_k c_k$$

$$\Rightarrow \underline{a}\cdot\underline{c}\ \underline{b}\cdot\underline{d} - \underline{a}\cdot\underline{d}\ \underline{b}\cdot\underline{c}$$

(ii) $e_{mp}e_{nq}\,\varepsilon_{mn,pq} = 0$

$$\Rightarrow (\delta_{mn}\delta_{pq} - \delta_{mq}\delta_{pn})\varepsilon_{mn,pq}$$

$$= \delta_{mn}\delta_{pq}\varepsilon_{mn,pq} - \delta_{mq}\delta_{pn}\varepsilon_{mn,pq}$$

$$= \varepsilon_{mm,pp} - \varepsilon_{qp,pq}$$

$$= \cancel{\frac{\partial^2\varepsilon_{11}}{\partial x_1^2}} + \cancel{\frac{\partial\varepsilon_{22}}{\partial x_2^2}} + \frac{\partial^2\varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} - \cancel{\frac{\partial^2\varepsilon_{11}}{\partial x_1^2}} - \frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2} - \cancel{\frac{\partial^2\varepsilon_{22}}{\partial x_2^2}} - \frac{\partial\varepsilon_{21}}{\partial x_2\partial x_1}$$

$$= \frac{\partial\varepsilon_{11}}{\partial x_2^2} + \frac{\partial\varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2} = 0$$

1. (b)(i) $\dot{\varepsilon}_{ij} = \dot{\lambda} \dfrac{\partial f}{\partial \sigma_{ij}}$, where $f = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ij}'} - \alpha \sigma_{kk} - Y$

Simplify by writing $\dfrac{\partial f}{\partial \sigma_{ij}} = \dfrac{\partial \bar{\sigma}}{\partial \sigma_{ij}} - \alpha \delta_{ij}$

and $\bar{\sigma}^2 = 3 J_2$ where $J_2 = \frac{1}{2}\sigma_{ij}'\sigma_{ij}'$

note that $\dfrac{\partial (\bar{\sigma}^2)}{\partial \sigma_{ij}} = 2\bar{\sigma}\dfrac{\partial \bar{\sigma}}{\partial \sigma_{ij}} \quad \therefore \dfrac{\partial \bar{\sigma}}{\partial \sigma_{ij}} = \dfrac{3}{2\bar{\sigma}} \dfrac{\partial J_2}{\partial \sigma_{ij}}$

Now, $\dfrac{\partial J}{\partial \sigma_{ij}} = \sigma_{pq}' \cdot \dfrac{\partial \sigma_{pq}'}{\partial \sigma_{ij}}$

$$\sigma_{pq}' = \sigma_{pq} - \frac{1}{3}\sigma_{kk} \cdot \delta_{pq}$$

$$\therefore \frac{\partial \sigma_{pq}'}{\partial \sigma_{ij}} = \delta_{pi}\delta_{qj} - \frac{1}{3}\delta_{ij}\delta_{pq}$$

$$\therefore \frac{\partial J}{\partial \sigma_{ij}} = \sigma_{pq}' \left(\delta_{pi}\delta_{qj} - \frac{1}{3}\delta_{ij}\delta_{pq} \right) = \sigma_{ij}'$$

(because $\sigma_{pq}'\delta_{ij}\delta_{pq} = \sigma_{pp}'\delta_{pp}\delta_{ij} = 0$)

$$\therefore \frac{\partial f}{\partial \sigma_{ij}} = \frac{3\sigma_{ij}'}{2\bar{\sigma}} - \alpha \delta_{ij}$$

$$\therefore \quad \dot{\varepsilon}_{ij} = \dot{\lambda} \left\{ \frac{3\sigma_{ij}'}{2\bar{\sigma}} - \alpha \delta_{ij} \right\}$$

(ii) from (i): $\dot{\varepsilon}_{ij} = \dot{\lambda}\left(\frac{3\sigma'_{ij}}{2\bar{\sigma}} - \alpha\delta_{ij}\right)$

squaring both sides and summing over the nine components of the tensor:

$$\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} = \dot{\lambda}^2\left(\frac{3}{2\bar{\sigma}}\sigma'_{ij} - \alpha\delta_{ij}\right)\left(\frac{3\sigma'_{ij}}{2\bar{\sigma}} - \alpha\delta_{ij}\right)$$

$$\Rightarrow \frac{3}{2}\dot{\bar{\varepsilon}}^2 = \dot{\lambda}^2\left(\frac{9\sigma'_{ij}\sigma'_{ij}}{4\bar{\sigma}^2} - \frac{3\sigma'_{ij}}{\bar{\sigma}}\alpha\delta_{ij} + \alpha^2\delta_{ij}\delta_{ij}\right)$$

$$= \dot{\lambda}^2\left(\frac{3}{2} + 3\alpha^2\right)$$

because $\sigma'_{ij}\delta_{ij} = 0$ and $\delta_{ij}\delta_{ij} = 3$

$$\therefore \dot{\lambda} = \frac{\dot{\bar{\varepsilon}}}{1+2\alpha^2}$$

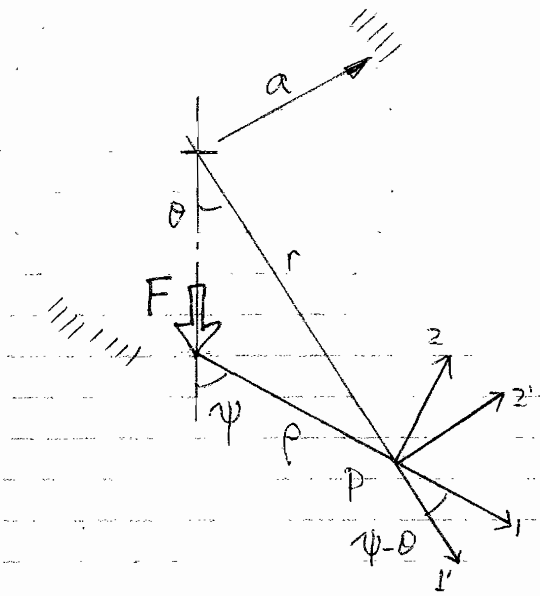
finally, $\frac{d\bar{\sigma}}{dt} = \frac{d\bar{\sigma}}{d\bar{\varepsilon}} \cdot \frac{d\bar{\varepsilon}}{dt}$, and for given flow rule,

$$\frac{d\bar{\sigma}}{d\bar{\varepsilon}} = nmk\exp(-n\bar{\varepsilon}) = n(k-\bar{\sigma})$$

$$\therefore \dot{\lambda} = \frac{\dot{\bar{\sigma}}}{(1+2\alpha^2)n(k-\bar{\sigma})}$$

2 答

(i)



Flamant Solution

$$\phi(\rho,\psi) = -\frac{F\rho\psi}{\pi}\sin\psi$$

from Table I of Data Sheet

$$\sigma_{\rho\rho} = \frac{-2F}{\pi\rho}\cos\psi, \quad \sigma_{\psi\psi} = \sigma_{\rho\psi} = 0$$

i.e. in (O12) coords

$$\sigma = \begin{bmatrix} \frac{-2F}{\pi\rho}\cos\psi & 0 \\ 0 & 0 \end{bmatrix}$$

(ii) Transfer to (r,θ) coords by rotation $(\psi-\theta)$ i.e. by 2-D rotation matrix

$$\begin{bmatrix} \cos(\psi-\theta) & -\sin(\psi-\theta) \\ \sin(\psi-\theta) & \cos(\psi-\theta) \end{bmatrix}$$

$$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$$

$$\sigma_{rr} = \sigma'_{11} = a_{1i} a_{1j} \sigma_{ij}$$

$$= a_{11}a_{11}\sigma_{11} + a_{12}a_{11}\sigma_{21} + a_{12}a_{11}\sigma_{12} + a_{12}a_{12}\sigma_{22}$$

$$\sigma'_{11} = \sigma_{rr} = \cos^2(\psi-\theta)\,\sigma_{\rho\rho}$$

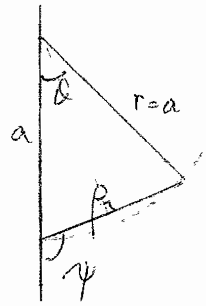
$$\sigma'_{22} = \sigma_{\theta\theta} = a_{2i}a_{2j}\sigma_{ij} = a_{21}^2\sigma_{11} = \sin^2(\psi-\theta)\,\sigma_{\rho\rho}$$

$$\sigma'_{12} = \sigma_{r\theta} = a_{1i}a_{2j}\sigma_{ij} = \cos(\psi-\theta)\sin(\psi-\theta)\,\sigma_{\rho\rho}$$

i.e. $\sigma_{rr} = \frac{-2F}{\pi\rho}\cos\psi\cos^2(\psi-\theta)$; $\sigma_{\theta\theta} = \frac{-2F}{\pi\rho}\cos\psi\sin^2(\psi-\theta)$

$$\sigma_{r\theta} = -\frac{2F}{\pi\rho}\cos\psi\cos(\psi-\theta)\sin(\psi-\theta)$$

iii) around edge of hole



$$\rho = 2a\sin\theta/2$$

$$\psi = \pi - \left(\tfrac{\pi}{2} - \theta/2\right)$$

$$\psi = \tfrac{\pi}{2} + \theta/2 \qquad \psi - \theta = \tfrac{\pi}{2} - \theta/2$$

$$\sigma_{rr} = -\frac{2F}{\pi}\,\frac{\cos(\pi/2+\theta/2)\cos^2(\pi/2-\theta/2)}{2a\sin\theta/2} = \frac{F}{\pi a}\sin^2(\theta/2)$$

$$\sigma_{rr} = \frac{F}{2\pi a}(1-\cos\theta) \qquad \begin{aligned}\cos(\pi/2+\theta/2) &= -\sin\theta/2\\ \cos(\pi/2-\theta/2) &= \sin\theta/2\end{aligned}$$

$$\sigma_{\theta\theta} = -\frac{2F}{2\pi a}\,\frac{\cos(\pi/2+\theta/2)\sin^2(\pi/2-\theta/2)}{\sin\theta/2} = \frac{F}{\pi a}\cos^2(\theta/2)$$

$$\sigma_{\theta\theta} = \frac{F}{2\pi a}(1+\cos\theta)$$

$$\sigma_{r\theta} = -\frac{2F}{2\pi a}\,\frac{\cos(\pi/2+\theta/2)\cos(\pi/2-\theta/2)\sin(\pi/2-\theta/2)}{\sin\theta/2} = \frac{F}{\pi a}\sin\theta/2\cos\theta/2$$

$$\sigma_{r\theta} = \frac{F}{2\pi a}\sin\theta$$

But around periphery of hole σ_{rr} must $\to 0$ so we need some more terms.

(iii) Stresses must be even functions of θ & decay as r increases: these limits form of $\phi(r,\theta)$ to $\ln r$, $r\theta\sin\theta$, $r\ln r\cos\theta$, $\cos\theta/r$ from Table I

So that $\phi(r,\theta)$ might contain

$$A\ln r + Br\theta\sin\theta + Cr\ln r\cos\theta + \frac{D\cos\theta}{r}$$

Table I

	$\phi(r,\theta)$	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
	r^2	2	2	0
	$r^2 \ln r$	$2\ln r + 1$	$2\ln r + 3$	0
A	$\ln r$	$1/r^2$	$-1/r^2$	0
	θ	0	0	$1/r^2$
	$r^3 \cos\theta$	$2r\cos\theta$	$6r\cos\theta$	$2r\sin\theta$
B	$r\theta\sin\theta$	$2\cos\theta/r$	0	0
C	$r\ln r\cos\theta$	$\cos\theta/r$	$\cos\theta/r$	$\sin\theta/r$
D	$\cos\theta/r$	$-2\cos\theta/r^3$	$2\cos\theta/r^3$	$-2\sin\theta/r^3$

$$r=a \left\{ \begin{aligned} \sigma_{rr} &= \frac{F}{2\pi a}(1-\cos\theta) + \frac{A}{a^2} + \frac{2B\cos\theta}{a} + \frac{C\cos\theta}{a} - \frac{2D\cos\theta}{a^3} \\ \sigma_{r\theta} &= \frac{F}{2\pi a}\sin\theta + \frac{C\sin\theta}{a} - \frac{2D\sin\theta}{a^3} \end{aligned} \right.$$

But both these must vanish

$$\left. \begin{aligned} &\frac{F}{2\pi a} + \frac{A}{a^2} = 0 \qquad A = \frac{-Fa}{2\pi} \\ &-\frac{F}{2\pi a} + \frac{2B}{a} + \frac{C}{a} - \frac{2D}{a^3} = 0 \\ \text{and } &\frac{F}{2\pi a} + \frac{C}{a} - \frac{2D}{a^3} = 0 \end{aligned} \right\} \text{ eqn ① to ③}$$

But these conditions insufficient to define A, B, C & D.

need to look at displacements in particular

$$\left\{ \begin{aligned} u_r(a,\theta) &= u_r(a,-\theta) \quad \text{symmetry} \\ u_\theta(a,\theta) &= -u_\theta(a,-\theta) \end{aligned} \right\} \text{ So use Table II}$$

No more than this reqd.

Now from ② & ③ $\quad \frac{2B}{a} = \frac{2F}{2\pi a} \quad \therefore \quad B = \frac{F}{2\pi}$

not required or expected

For plane strain $\kappa = 3 - 4\nu$; for planes stress $\kappa = (3-\nu)/(1+\nu)$

	$\phi(r,\theta)$	$2Gu_r$	$2Gu_\theta$
	r^2	$(\kappa-1)r$	0
	$r^2 \ln r$	$(\kappa-1)r\ln r - r$	$(\kappa+1)r\theta$
A	$\ln r$	$-1/r$	0
	θ	0	$-1/r$
	$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
B	$r\theta\sin\theta$	$0.5[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\ln r\cos\theta]$	$0.5[(\kappa-1)\theta\cos\theta + \sin\theta - (\kappa+1)\ln r\sin\theta]$
C	$r\ln r\cos\theta$	$0.5[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\ln r\cos\theta]$	$0.5[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\ln r\sin\theta]$
D	$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$

$$2Gu_r = -\frac{A}{r} + \frac{B}{2}\left[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\ln r\cos\theta\right] + \frac{C}{2}\left[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\ln r\cos\theta\right] + \frac{D\cos\theta}{r^2}$$

for symmetry all ok

$$2Gu_\theta = \frac{B}{2}\left[(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\ln r\sin\theta\right] + \frac{C}{2}\left[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\ln \sin\theta\right]$$

But also $(u_\theta)_0 = (u_\theta)_{2\pi} = 0$

$$\frac{B}{2}\left[(\kappa-1).2\pi\right] + \frac{C}{2}\left[(\kappa+1).2\pi\right] = 0$$

$$B(\kappa-1) = -C(\kappa+1)$$

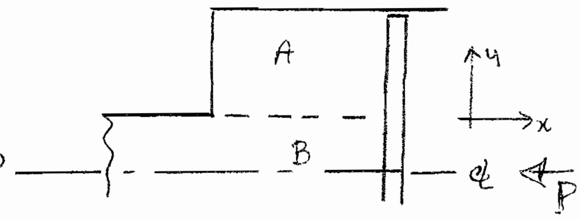
$$\therefore\ C = -\frac{\kappa-1}{\kappa+1}\cdot\frac{F}{2\pi}$$

and from ③ $\dfrac{2D}{a^3} = \dfrac{F}{2\pi a} - \dfrac{F}{2\pi a}\dfrac{(\kappa-1)}{(\kappa+1)} \Rightarrow D = \dfrac{Fa^2}{2\pi(\kappa+1)}$

3(a) Choosing the simplest stress field, suppose

within region A $\begin{cases} \sigma_{xx} = Y = 2k \\ \sigma_{yy} = 0 \end{cases}$

and region B $\sigma_{xx} = \sigma_{yy} = 0$



Now Lower Bound theorem $\int_{\Gamma_v} \underline{t} \cdot d\underline{u}\, dS \geq \int_{\Gamma_v} \underline{t}^* \cdot d\underline{u}\, dS$

Γ_v is the boundary on which velocities are prescribed – in this case these are only nonzero at the ram, when $\underline{t}^* \cdot d\underline{u} = 2k \cdot v\, dt$ in region A and zero in region B.

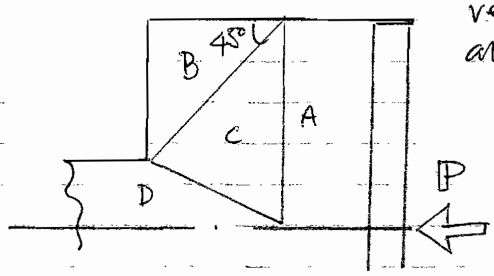
$$v\,\delta t \int_{\Gamma_v} \underline{t} \cdot \underline{n}\, dS \geq 4h \cdot 2k v\, \delta t$$

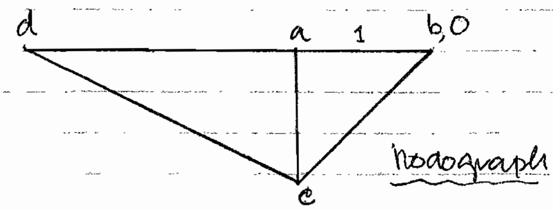
taking account of both halves of process

But $\int \underline{t} \cdot \underline{n}\, dS = P$ ram force

$$P \geq 8hk$$

(b) Choose shear lines as shown – for simplicity let Dead Metal Zone B have 45° angles*, and boundary between regions A and C meet centre line at 90°





Upper Bound Theorem

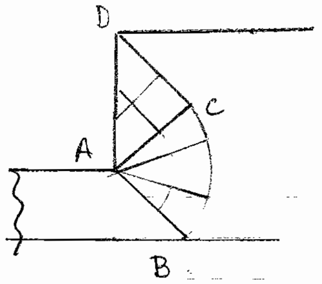
$$\int_{\Gamma_v} \underline{t} \cdot d\underline{u}\, dS \leq \sum_{\substack{i\ \text{shear} \\ \text{planes}}} k\, L_i\, |v_i^{\text{rel}}|\, dt$$

thus $P \leq 2k\{l_{ac} v_{ac} + l_{bc} v_{bc} + l_{cd} v_{cd}\}$

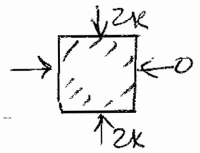
$$P \leq 2k(3h \cdot 1 + 2\sqrt{2}h \cdot \sqrt{2} + \sqrt{5}h \cdot \sqrt{5})$$

ie. $\underline{P \leqslant 24 \text{ kN}}$ (*in fact this is the geometry that minimises P see Calladine Engineering Plasticity 3.10.7)

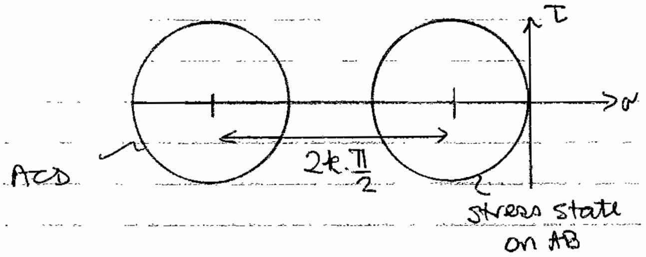
(c)



There is no force to the left of line AB, so the stress-state of an element on that line is



Moving along slipline BCD clearly moving into a compressive region, so in terms of Mohr's circle.



$p_{AB} = k$

Moving from B to C, $p_{AC} = p_{AB} + 2k\phi$

where $\phi = \pi/2$ $\therefore$ $\underline{p_{AC} = k + 2k.\pi/2}$

Within ACD slip-lines are straight so no change in pressure - all material has same Mohr's circle, ie. $p_{AD} = p_{AC}$

Assuming no friction along AD, so that slip lines meet surface at 45° then

$$\sigma_{xx} = p_{AD} + k\sin(2\times 45°)$$

$$\sigma_{xx} = k + k.\pi + k = (2+\pi)k$$

Now by horizontal equilibrium

$$(2+\pi)k.\,4h = P$$

$$\underline{P = 20.57 \text{ kN}}$$

(d) All three methods have assumed rigid-plastic material behaviour and frictionless contact with wall of container - probably far from reality - and in addition a real process would not have infinite width.

However, both the U.B. and S.L.F methods have assumed plausible deformation or collapse mechanisms whereas the LB solution used a rather implausible stress field. It is reasonable to assume that the actual ram load will be closer to the other two estimates - say ca. 20 kN.

Examiner's Comments

Q1 Part (a) almost universally correctly done. Part (b) - more mixed but with several substantially correct.

Q2 Several candidates failed to use the information provided in Table I of the Data sheet to simply write down $\sigma_{\rho\rho}$, $\sigma_{\varphi\varphi}$ and $\sigma_{\rho\varphi}$. Nevertheless the question with the highest average mark.

Q3 The Bound theorems were well stated but surprisingly poorly applied. Several candidates varied the point of intersection of the shear planes on the centre line in order to find the 'best' Upper Bound estimate. The Slip Line Field element was not well done.