4F1 2008 Solutions

1(a) Advantages: reduction of effect of disturbances (and noise)
reduction of sensitivity to plant uncertainty
robustness of stability " " "
reduction of effects of non-linearity
stabilisation of unstable systems

Disadvantages: sensitivity can be worsened in some freq. ranges
noise can be amplified " " " "
feedback is expensive to implement
modelling, design and commissioning of a control system are time-intensive and expensive

(b) (i) Bookwork (see attached).

(ii) $$|1+G(j\omega)k(j\omega)| \geq 1 - |G(j\omega)k(j\omega)|$$

Hence $$|S(j\omega)| \leq \frac{1}{1-|G(j\omega)k(j\omega)|}$$

providing $|G(j\omega)k(j\omega)| \leq 1$.

$$\Rightarrow \ln|S(j\omega)| \leq -\ln(1-\omega^{-2})$$

for $\omega \geq 5$

Consider $$\int_5^\infty \ln(1-\omega^{-2})\,d\omega = \left[\omega\ln(1-\omega^{-2}) + \ln\left(\frac{\omega+1}{\omega-1}\right)\right]_5^\infty$$

$$= -5\ln\frac{24}{25} - \ln\frac{3}{2}$$

$$= -0.201$$

(Note that $\omega\ln(1-\omega^{-2}) \simeq \omega.\omega^{-2} \to 0$ for large ω)

$$0 = \int_0^\infty \ln |S(j\omega)| \, d\omega$$

$$\leq \int_0^1 \ln \varepsilon \, d\omega + \int_1^5 \ln 1.2 \, d\omega + 0.201$$

$$= \ln \varepsilon + 4 \ln 1.2 + 0.201$$

$$\Rightarrow \quad \varepsilon \geq 0.394$$

1(b)(i) Let L denote the return ratio which has at least 2nd order roll-off at high frequency by assumption. $S(s)$ is analytic and has no zeros in RHP so $\log S(s)$ is analytic in $Re(s) > 0$. Let $\sigma = \sigma_0$ and $\omega_0 = 0$ in the Poisson formula (data sheet). Then:

$$\sigma \ln |S(\sigma)| = \frac{2}{\pi} \int_0^\infty \frac{\sigma^2}{\sigma^2 + \omega^2} \ln |S(j\omega)| d\omega$$

As $\sigma \to \infty$ the RHS converges to:

$$\frac{2}{\pi} \int_0^\infty \ln |S(j\omega)| d\omega$$

This follows basically because $\frac{\sigma^2}{\sigma^2+\omega^2}$ is close to one except for large ω, and $\ln |S(j\omega)| \to 0$ as $\omega \to \infty$ because $|S(j\omega)| \to 1$ as $\omega \to \infty$.

By assumption

$$L(\sigma) \sim \frac{c}{\sigma^k}$$

for large σ where $k \geq 2$, and c is a real constant. Thus:

$$\begin{aligned}
\sigma \ln |S(\sigma)| &\sim -\sigma \ln(1 + c\sigma^{-k}) \\
&= -\sigma(c\sigma^{-k} + \ldots) \\
&= -c\sigma^{-k+1} + \ldots
\end{aligned}$$

Thus

$$\sigma \ln |S(\sigma)| \to 0$$

as $\sigma \to \infty$. We therefore obtain:

$$\int_0^\infty \ln |S(j\omega)| d\omega = 0$$

2(a)(i) Write $G = \frac{N_G}{D_G}$, $K = \frac{N_K}{D_K}$, $F = \frac{N_F}{D_F}$, $H = \frac{N_H}{D_H}$, where numerator and denominator polynomials have no common factor and $\deg(N_G) < \deg(D_G)$, $\deg(N_K) \leq \deg(D_K)$, $\deg(N_F) \leq \deg(D_F)$, $\deg(N_H) \leq \deg(D_H)$. We can now write:

$$T_{\bar{r}\to\bar{y}} = \frac{GKH}{1+GKF} = \frac{N_G N_K D_F}{N_G N_K N_F + D_G D_K D_F} \cdot H$$

Inspection of the transfer function reveals two necessary conditions:

(a) Any RHP zero of G must remain in $T_{\bar{r}\to\bar{y}}$,

(b) $T_{\bar{r}\to\bar{y}}$ must roll off at high frequencies at least as fast as G.

(a)(ii) Note that RHP roots of N_G cannot be cancelled by roots of $N_G N_K N_F + D_G D_K D_F$ or by poles of H which gives (a). To see (b), note that the high frequency Bode slope of $T_{\bar{r}\to\bar{y}}$ is -20 dB multiplied by:

$$\deg\left(N_G N_K N_F + D_G D_K D_F\right) - \deg\left(N_G N_K D_F\right) + \deg(D_H) - \deg(N_H)$$

which in turn is

$$\begin{aligned} &\geq \deg\left(D_G D_K D_F\right) - \deg\left(N_G N_K D_F\right) \\ &= \deg\left(D_G\right) - \deg\left(N_G\right) + \deg\left(D_K\right) - \deg\left(N_K\right) \\ &\geq \deg\left(D_G\right) - \deg\left(N_G\right). \end{aligned}$$

(a)(iii) L is given, i.e. we know the product $KF = C$. Let us split C as follows.

(i) Put any RHP zeros of C (unlikely to be any) into F,

(ii) Put any RHP poles of C into K,

(iii) Ensure $\deg(N_K) = \deg(D_K)$. This may need the introduction of cancelling LHP poles and zeros into K and F.

This makes $N_K D_F$ a stable polynomial. If R denotes the desired transfer function $T_{\bar{r}\to\bar{y}}$ then we can find H from the formula:

$$H = R \cdot \frac{N_G N_K N_F + D_G D_K D_F}{N_G N_K D_F}.$$

Note that H is stable with bounded high frequency gain.

2(b)

$$F(s)\,G(s) = \frac{s-1}{(s+1)(s-2)}$$

Try $\quad K(s) = \dfrac{k(s+1)}{s-p}$

Closed-loop poles: $\quad k(s-1) + (s-2)(s-p)$

$$\equiv \quad s^2 - (2+p-k)\,s + 2p - k$$

Need: $\quad \begin{array}{l} 2p-k > 0 \\ k-2-p > 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} 2p > k \\ k > 2+p \end{array}$

e.g. $\quad p = 4\,, \quad k = 7$

$$\frac{KG}{1+PKG} = \frac{\dfrac{7(s+1)}{s-4}\;\dfrac{1}{s-2}}{1 + \dfrac{7(s-1)}{(s-2)(s-4)}}$$

$$= \frac{7(s+1)\cancel{(s-2)}}{s^2+s+1}$$

$$\Rightarrow \quad H(s) = \frac{s^2+s+1}{7(s+1)^2}$$

(ii) $$FGK = \frac{\cancel{s-1}}{\cancel{s+1}} \frac{1}{s-2} k \frac{7(s+1)}{s-4}$$

$$= \frac{7k(s-1)}{(s-2)(s-4)}$$

c.l. poles $\quad s^2 + (7k-6)s + 8 - 7k$

$$7k - 6 > 0 \qquad k > \frac{6}{7}$$

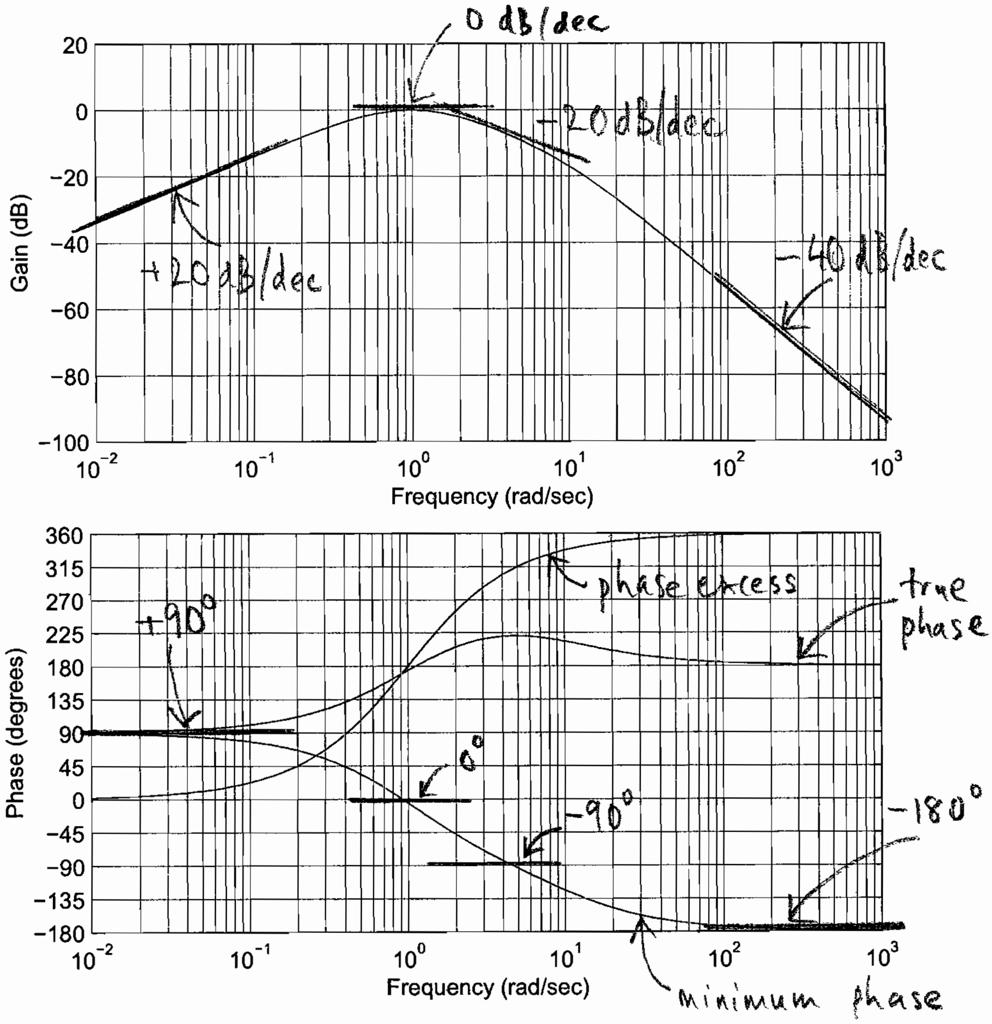
$$8 - 7k > 0 \qquad k < \frac{8}{7}$$

v. small allowed range

(iii) Control system has poor robustness properties.

Advice: try to find a sensor to measure y directly and avoid the need for an unstable controller.

3. (a)



(i) Straight-line approximations to magnitude allow approximate min. phase to be estimated using Bode gain-phase relationships. Subtracting gives phase excess.

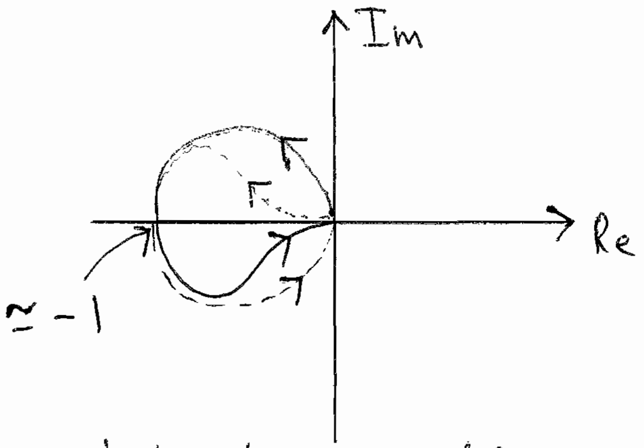
(ii) The +20 dB/dec slope at low frequencies and +90° phase suggest a (single) zero in $G(s)$ at $s=0$.

(iii) Phase excess rises from zero to +360°, suggesting two RHP poles. Phase excess $\simeq 180°$ at 1 rad/sec suggests two poles around 1 rad/sec.

(iv) For a conventional loop shape, crossover frequency should be at least 1 rad/sec.

[Actual transfer function: $G(s) = \dfrac{20s}{(s-1)^2(s+10)}$ not needed.]

(b) (i)



At ω slight greater than 1 rad/sec $\measuredangle G(j\omega) = -180°$ and $|G(j\omega)|$ is (slightly less than) 0 dB.

Two anticlockwise encirclements required for closed-loop stability.

2 RHP poles $\Leftrightarrow$ $-\frac{1}{k} < -1$ or $-\frac{1}{k} > 0 \Leftrightarrow k < 1$

0 RHP poles $\Leftrightarrow$ $-1 < -\frac{1}{k} < 0 \Leftrightarrow 1 < k < \infty$

(approx.)

(ii) Largest phase occurs around $\omega = 4.5$ rad/sec
Magnitude ~ -7 dB at 4.5 rad/sec $\Rightarrow$ $k \simeq 2.2$

(c) Since $G(s)$ has a zero at $s = 0$, $k(s)$ cannot have a pole there (for internal stability) hence

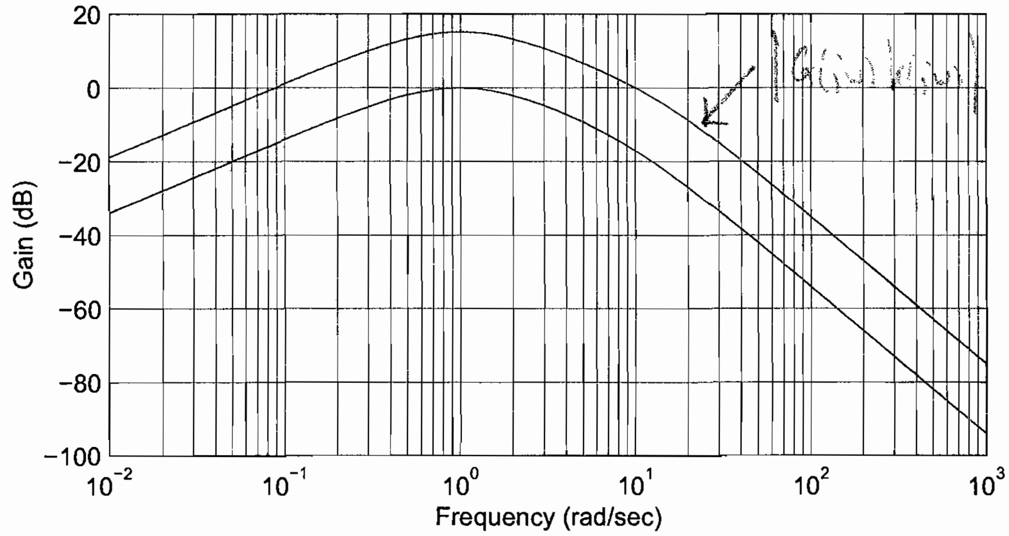
$$S(0) = \frac{1}{1 + G(0)k(0)} = 1$$

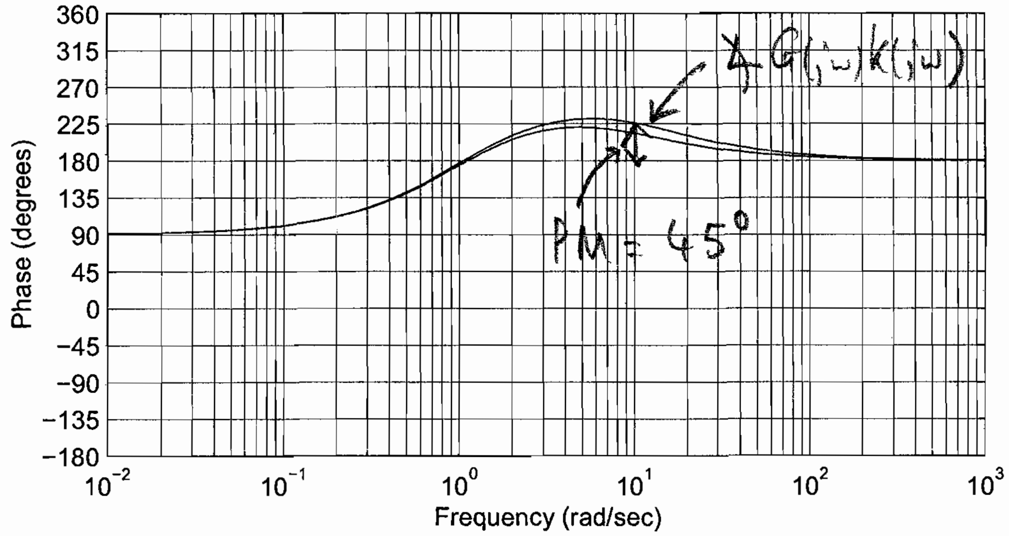
for any stabilising (internally) controller.

(d) At $\omega = 10$ rad/sec, $\measuredangle G(j\omega) \simeq 215°$, so we only need to increase phase by about $10°$ and adjust the crossover to achieve both specs. A lead compensator will suffice, e.g.

$$k(s) = 7.15 \times 1.25 \; \frac{s + 10/1.25}{s + 10 \times 1.25}$$

(4) cont.


Gain (dB)
20
0
−20
−40
−60
−80
−100
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}
Frequency (rad/sec)


Phase (degrees)
360
315
270
225
180
135
90
45
0
−45
−90
−135
−180
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}
Frequency (rad/sec)
$\angle G(j\omega)K(j\omega)$
PM = 45°